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13. ABSTRACT *(Maximum 200 words)*

General Description of Program: The principal investigator, his associates and research assistants have carried out a program of research in the general area of distributed parameter systems modelling, identification and control. This work has been concentrated in four main areas. (1) Control of the Korteweg - de Vries and other nonlinear partial differential equations exhibiting solitary waves. (2) Frequency domain analysis of distributed parameter systems. (3) Energy dissipation mechanisms in elastic structures. (4) Modelling and parameter estimation in elastic structures.

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MODELLING, INFORMATION PROCESSING, AND CONTROL

FINAL REPORT ON ACTIVITIES SUPPORTED BY

AFOSR 89 - 0031

Prepared by

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Submitted March 15, 1992

FINAL REPORT ON ACTIVITIES SUPPORTED BY

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1. <u>General Description of the Program</u> Commencing November 1, 1988, and continuin to the present, the principal investigator, his associates and research assistants have carried out a program of research in the general area of distributed parameter systems modelling, identification and control. This work has been concentrated in four main areas:

i) Control of the Korteweg - de Vries and other nonlinear partial differential equations exhibiting solitary waves;

ii) Frequency domain analysis of distributed parameter systems;

iii) Energy dissipation mechanisms in elastic structures;

iv) Modelling and parameter estimation in elastic structures.

The research program has included mathematical studies by the principal investigator, contributions from visiting associate researchers and dissertations prepared by PhD students of the principal investigator. Related activities include experimental work in the MIPAC-VT laboratory by the principal investigator and visiting associates, presentation of results at scientific meetings and preparation and publication of research articles. In the sequel we first provide brief descriptions of each area of research and the results obtained and then summarize research publication activity, visitors, scientific meetings, etc.

2. Brief Descriptions of Research Areas and Results

i) Control of the Korteweg - de Vries and other nonlinear partial differential equations exhibiting solitary waves.

With the control theory of linear partial differential equations having now reached a certain state of maturity, we have felt it to be important to initiate studies for nonlinear systems with significant applications. As in the case of the early development of the control theory for linear distributed parameter systems, we have found it desirable to focus on nonlinear systems for which a considerable body of theory exists in regard to the classical questions of existence, uniqueness and regularity before in order to obtain definitive control results. At the present time we are studying two nonlinear dispersive wave equations, the Korteweg - de Vries (KdV) equation and the related Boussinesq equation and a nonlinear diffusion equation, the Benjamin - Bona - Mahony (BBM) equation. A number of important positive results have been obtained in the KdV and BBM cases. It is significant that the study of these equations indicates questions for related linear systems which otherwise might be ignored. Here we will describe results obtained for the KdV equation; research is in preliminary stages for the other two.

Work carried out jointly with Professor B.-Y. Zhang of the University of Cincinnati and Prof. V. Komornik of the University of Strasbourg, France, involves the forced KdV equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = f \qquad (1.01)$$

on the region $0 \leq x \leq 2\pi$, $t \geq 0$, with periodic boundary conditions

$$w^{(k)}(2\pi,t) = w^{(k)}(0,t), \quad k = 1,2,3, \ldots ,$$

so that $x = 2\pi$ is effectively identified with $x = 0$.

For the unforced equation ($f(x,t) \equiv 0$) it is well known that there is an infinite set of conserved quantities. Only the first three of these are of interest to us:

$$\int_0^{2\pi} w(x,t)\ dx\ , \tag{1.02}$$

$$\int_0^{2\pi} w(x,t)^2\ dx\ , \tag{1.03}$$

$$\int_0^{2\pi} \left[\left[\frac{\partial w}{\partial x}(x,t)\right]^2 - \frac{1}{3}\ w(x,t)^3\right]\ dx\ . \tag{1.04}$$

Much of our control analysis involves examination of the behavior of these quantities when the forcing term, $f(x,t)$, is not identically zero. For suitably smooth solutions of (1.01) we have

$$\frac{\partial}{\partial t}\int_0^{2\pi} w(x,t)\ dx = \int_0^{2\pi} \left[-w(x,t)\frac{\partial w}{\partial x}(x,t) - \frac{\partial^3 w}{\partial x^3}(x,t) + f(x,t)\right]dx =$$

$$= \int_0^{2\pi} \left[\frac{\partial}{\partial x}\left[-\frac{1}{2}w(x,t)^2 - \frac{\partial^2 w}{\partial x^2}(x,t)\right] + f(x,t)\right]\ dx = \int_0^{2\pi} f(x,t)\ dx\ ,$$

from which we conclude that "volume" is conserved if we require

$$\int_0^{2\pi} f(x,t)\ dx = 0\ . \tag{1.05}$$

In fact we are interested in a particular sort of control action. Let $I = [a,b]$ be a subinterval of $[0,2\pi]$, require

$$f(x,t) \equiv 0\ ,\ x \notin I\ ,$$

and determine f on I by

$$f(x,t) = -k(w(x,t) - [w]_I)\ , \tag{1.06}$$

where k is a positive "gain" constant and

$$[w]_I = [w]_I(t) = \frac{1}{b-a}\int_a^b w(x,t)\,dx$$

is the mean value of w over I. It is trivial to verify that (1.05) is satisfied. This sort of control can be realized, approximately, by a distributed "pumping" action over the interval I which injects "fluid" into the system at points x where $w(x,t) < [w]_{[a,b]}(t)$ and extracts "fluid" from the system where the opposite inequality obtains, at a proportional rate determined by k in each case.

The feedback control law just described is intended to stabilize the controlled system in a certain sense. Since volume is conserved, for a solution of (1.01) with initial state

$$w(x,0) = w_0(x)$$

we have, for all t,

$$V(w(\cdot,t)) = \int_0^{2\pi} w(x,t)\,dx = V(w_0) = \int_0^{2\pi} w(x,0)\,dx\ ;$$

equivalently, abbreviating $[w]_{[0,2\pi]}$ by $[w]$ for convenience,

$$[w](t) \equiv [w_0]\ .$$

We then note that for any $w \in L^2[0,2\pi]$ satisfying this identity

$$\int_0^{2\pi} w(x,t)^2\,dx = \int_0^{2\pi}\left\{w(x,t) - [w] + [w]\right\}^2 dx =$$

$$\int_0^{2\pi}\left\{w(x,t) - [w]\right\}^2 dx + \underbrace{2[w]\int_0^{2\pi}\left\{w(x,t) - [w]\right\} dx}_{=0} + \int_0^{2\pi} [w]^2\,dx$$

$$= \int_0^{2\pi} \left[w(x,t) - [w]\right]^2 dx + \int_0^{2\pi} [w]^2 \, dx .$$

Hence, among functions w for which $V(w) = V(w_0)$, the integral (1.03) is minimized by the constant function [w]. We may therefore hope to cause w(x,t) to approach [w] as $t \to \infty$ by using a control law designed so that this integral is non-increasing.

Since the KdV equation has the property of invariance under time reversal (replace x by -x at the same time as we replace t by -t), global controllability results for the KdV equation are in our grasp if we can use the LaSalle Invariance Principle to obtain global asymptotic stability of the zero equilibrium state and if we can obtain local controllability results in a neighborhood of that state.

The application of the LaSalle Invariance Principle referred to above requires, for its implementation, certain unique continuation results for solutions of the unforced KdV equation. We have, in fact, shown that any nontrivial solution of the KdV equation lying in an appropriate function space ($L^\infty(-\infty,\infty)$ in this case) cannot have support restricted to two left (or right) horizontal lines in x-t space. This result implies that the solution u cannot have compact support at two different times and also implies that if the solution u vanishes in an open set in x-t space, then it vanishes everywhere. It turns out that these results provide the proof of triviality of certain invariant sets for the closed loop KdV system described above which is essential for completion of the asymptotic stability result. In order to complete the global asymptotic stability part of the work it is necessary to show that solutions of the controlled system lie in a compact subset of the state space, $L^2[0,2\pi]$. This part of the work has proved remarkably recalcitrant and requires our continuing strenuous efforts for successful completion.

The required local controllability results for the nonlinear equation in a neighborhood of $w \equiv 0$ follow from exact controllability results for the linearized system, which is just the third order linear dispersion equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} = f .$$

We have completed a thorough study of the uniform exponential stabilizability and exact controllability of this system and a related boundary control system with positive results in each instance. This work is complete and will appear as a paper jointly authored by B.-Y. Zhang and the principal investigator. The results on the KdV equation are currently being prepared as an article for publication jointly authored by V. Komornik, B.-Y. Zhang and the principal investigator.

ii) Frequency domain analysis of distributed parameter systems.

In this area we have been concerned with the development of frequency domain methods useful for analyzing hyperbolic systems corresponding to various types of wave motion, including vibration of elastic systems of various types. The transfer functions and related inout - output operators corresponding to these types of systems have some disconcerting properties, particularly in relation to convergence questions with respect to approximation via finite dimensional systems. In addition to these approximation questions we have been studying the use of transfer function methods to analyze closed loop systems arising out of certain linear feedback laws in distributed parameters systems, particularly where boundary feedback is concerned, use of transfer functions to determine the admissibility of various input and output mechanisms and a number of other related matters. In this process we have developed a representation of closed loop semigroups in terms of the inverse Laplace transforms of the associated closed loop transfer functions, providing what we believe to be a significantly new approach to certain problems in this area.

Our work in this area has been greatly strengthened by the presence of Dr. G. Weiss as a visiting researcher. He has developed the particularly useful category of *regular* linear systems which are, to put it briefly, linear input-output systems whose response to an impulse input occurring at $t = 0$ has a certain limiting behavior as $t \downarrow 0$. The introduction of this class of systems and the systematic development of its properties has clarified many previously obscure questions.

Dr. Weiss has been investigating the connection between state space and transfer function representations of infinite dimensional

linear systems. The central question is to find verifiable conditions for a system to be regular, since regular systems are known to have a convenient state space representation.

Dr. Weiss has been working with the principal investigator in some research on well posedness of feedback systems when the feedback operator is unbounded. Conditions have been found for the closed loop system to be regular. Explicit characterizations of the closed loop generator, control operator and observation operator have been obtained. More recent work with the Principal Investigator has been focussed on the question of a frequency domain characterization of exact controllability and/or observability of infinite dimensional linear systems, extending to infinite dimensions a criterion due to M. L. Hautus in the finite dimensional setting. This research is now complete and has been submitted for publication in the SIAM Journal on Control and Optimization as a joint paper by G. Weiss and the principal investigator.

Dr. Weiss has been cooperating with the principal investigator's former PhD student, Dr. Scott Hansen, on extension of the Carleson measure criterion for admissibility of control operators to the case when the input space is infinite dimensional. They have also investigated the connection between admissibility and the related operator Lyapounov equation.

Working with Prof. Richard Rebarber of the University of Nebraska, Dr. Weiss has found some general conditions for a feedback system to be robustly stable with respect to small delays in the feedback loop. Specifically, they have found that to ensure robustness, the open loop transfer function should decay in all directions in the right half complex plane at a certain rate. This research is also complete and will appear as a paper jointly authored by Dr. Weiss and Prof. Rebarber.

iii) Energy dissipation mechanisms in elastic structures.

Our research program in this area has been concentrated on the study of internal dissipation mechanisms in the context of the Euler-Bernoulli beam equation

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^4}\right] = 0 \qquad (3.01)$$

for which, in the absence of additional dissipative terms, the energy

$$\mathcal{E}\left(w,\frac{\partial w}{\partial t}\right) \equiv \frac{1}{2}\int_0^L\left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx \qquad (3.02)$$

is conserved. Difficulties encountered in developing mathematical models for energy dissipation in this system agreeing with dissipation rates measured in the MIPAC-VT laboratory led earlier to the development of an approach which, accurately or not, has been named the *spatial hysteresis* damping model. This model consists, primarily, in the integro-partial differential equation

$$\rho\frac{\partial^2 w}{\partial t^2} - 2\frac{\partial}{\partial x}\int_0^L h(x,\xi)\left[\frac{\partial^2 w}{\partial t\partial x}(x,t) - \frac{\partial^2 w}{\partial t\partial x}(\xi,t)\right]d\xi + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0 \qquad (3.03)$$

involving an appropriate *interraction kernel* $h(x,\xi)$. The analytically derived damping rates associated with this model show excellent agreement with laboratory measurements over a wide frequency range, as described in reports filed earlier.

Damping mechanisms such as this one are *direct mechanisms* in that they involve direct insertion of supplementary dissipation terms into the original conservative equations governing the elastic system. Mechanisms studied in our recently completed work are *indirect* in the sense that they involve coupling the mechanical equations governing beam motion to related dissipative systems with additional dynamics, resulting in an overall system in which mechanical energy is dissipated. We have carefully analyzed the Euler-Bernoulli beam with *thermoelastic* damping and with *shear diffusion* damping. The most significant situation is that wherein both types of damping are simultaneously present. The paper embodying this research has been completed and has been accepted for publication in the Quarterly of Applied Mathematics. It will appear shortly.

Indirect damping mechanisms enjoy the advantage that they are motivated and derived from simple physical considerations, allowing us to answer objections often made concerning the spatial hysteresis

model to the effect that it is ultimately *ad hoc* in nature. Because of these desirable properties we have thought it useful to study such mechanisms from a fairly general viewpoint. In an article just completed we have introduced a general analytic and algebraic structure within which many such indirect damping mechanisms, including but not restricted to the thermal and shear diffusion mechanisms, may be incorporated. The resulting framework is sufficiently general to indicate the possibility of many other indirect damping mechanisms than those already dealt with and precise enough so that it permits us to see that thermal damping, even when longitudinal heat conduction is admitted, is mathematically, as well as physically, quite distinct from shear diffusion damping. This article has been submitted to the Journal of Mathematical Analysis and Applications.

In this area we have also been concerned with the development of new laboratory techniques which will allow us to overcome experimental difficulties long encountered in separating the effects of internal damping from those due to interaction with the supporting apparatus and/or measuring devices. A new technique has now been developed allowing one to study waves in the elastic structure as they leave the supporting clamp structure, comparing them with waves returning to that structure after reflection from the free end of the elastic beam under study. This technique uses MIPAC-VT's laser vibrometer to eliminate effects due to measurement devices, as are common with the use of accelerometers and/or stress gauges, and relies on the mathematically provable fact that damped modes in an elastic structure do not have true *nodes* as developed theoretically for undamped structures. Internal damping rates are inferred from comparison of vibration amplitudes at *quasi-nodes* near the clamping mechanism with corresponding quasi-nodal amplitudes near the free end of the structure. It is these easily made observations which allow experimental assessment of the outgoing and incoming wave amplitudes as discussed earlier. A paper on this experimental technique, along with some experimental results, is currently in preparation.

iv) Modelling and parameter estimation in elastic structures.

Here we are referring to the work described under section c) of the Renewal Proposal to which this appendix is attached. We will not repeat the basic discription of the research program as developed there. Rather, we will indicate what work has been carried out up to

the present time with the support of AFOSR 89 - 0031. This work has been a cooperative undertaking with Professor Luther White of the University of Oklahoma, whose research is also partially supported by AFOSR.

Extensive use of the research equipment in the MIPAC-VT facility has been made by the principal investigator, Assistant Professor Robert C. Rogers, and Luther White, visiting from the University of Oklahoma, in study of modal frequencies, shapes and damping rates in a variety of elastic structures. Major efforts have been made to improve laboratory capabilities in this area with the design and installation of equipment allowing separate excitation of lateral and torsional modes in axially symmetric narrow plates. This equipment was first put to use during a spring 1991 visit by Luther White, with experimental results far better than any obtained previously. This data is currently being analyzed for identification purposes.

This experimental data is complemented by our efforts to develop usable narrow plate models, starting with the Mindlin - Timoshenko plate model. Joint work by the principal investigator and Prof. Luther White has now resulted in such a model which appears to offer real advantages from the identification standpoint. This model is a distributed model insofar as the longitudinal axis of the narrow plate is concerned but is finite dimensional as far as description of shearing and torsional motions orthogonal to the longitudinal axis is concerned. Mathematically, the resulting model is a linear symmetric hyperbolic system; an extension of the familiar Timoshenko beam equations. It is far simpler to work with than the two dimensional plate equations and, in that respect, appears to offer real computational advantages for identification purposes.

3. Research Associates and Assistants Supported in Part by AFOSR 89 - 0031

In accordance with the provisions of the subject grant, three visiting research associates were supported during the grant period.

Dr. Gunter Leugering of Technische Hochschule Darmstadt, Darmstadt, Germany, was supported in part by grant funds during the period January 1, 1989 through April 30, 1989 as a visiting Assistant Professor in the Mathematics Department at Virginia Tech. Dr. Leuger-

ing assisted the principal investigator in development of the shear diffusion damping model and the related projection method. He also carried out research in control theory of partial differential equations with delay terms, such as arise, for example, in connection with elastic systems with viscoelastic damping.

Grant funds were used during the period covered by the grant to provide partial support for Dr. George Weiss, who visited Virginia Tech from the Weizmann Institute, Rehovoth, Israel. It was sufficient to underwrite only 20% of Dr. Weiss salary as the larger part was provided in the form of a Weizmann Fellowship. Dr. Weiss carried out an intensive program of research in the area of frequency domain methods for infinite dimensional linear systems. His appointment extended from December 15, 1989 through December 14, 1990.

Our research efforts in nonlinear distributed parameter control were greatly aided by the visit to Virginia Tech, during Sem. I, AY 1990-91, by Professor V. Komornik, newly appointed Professor of Mathematics at the University of Strasbourg, France. Professor Komornik, who was partially supported by this AFOSR grant, has gained world recognition for his exceptionally insightful treatment of control problems for wave and elastic systems, as well as certain nonlinear distributed parameter systems. He worked with the principal investigator and the principal investigator's former PhD student, Dr. B.-Y. Zhang on global controllability of the Korteweg - de Vries equation.

The principal investigator's former PhD. student, Scott Hansen, was, in part, supported by the grant from November 1, 1988 through April 30, 1989. Mr. Hansen had also been supported in part by AFOSR at the University of Wisconsin from an earlier grant at that University. In December of 1989 Mr. Hansen received the Mathematics PhD at the University of Wisconsin - Madison, presenting a thesis titled *Frequency - Proportional Damping Mechanisms in Elastic Beams*. This thesis dealt with mathematical properties of the spatial hysteresis damping mechanism and related "bending rate" models, with particular attention being given to the properties of the strongly continuous semigroups associated with such models and the character of the generating operator of the semigroup. Dr. Hansen continued these researches at Virginia Tech during Sem. II, 1988-89. He is currently at Iowa State University.

The principal investigator's former PhD student, Mr. B. - Y.

Zhang, received his PhD from the University of Wisconsin, Madison, in May, 1990, presenting a thesis titled *Some Results for Nonlinear Dispersive Wave Equations with Application to Control*. Although Mr. Zhang was not directly supported by grant funds, his thesis research nevertheless parallelled and supported the principal investigator's work in the control of nonlinear partial differential equations. Dr. Zhang is now Assistant Professor at the University of Cincinnati.

4. Research Articles Resulting from Work Supported in Part by AFOSR 89 - 0031

The following is a list of papers reporting research supported in part by AFOSR 89 - 0031. In the case of papers by the principal investigator it includes some articles for which the research was carried out under the aegis of earlier AFOSR grants but for which processing continued under AFOSR 89 - 0031. Listed papers by other authors report only research carried out under AFOSR 89 - 0031.

Papers by the principal investigator:

On mathematical models for the elastic beam with frequency proportional damping, in "*Control and Estimation in Distributed Parameter Systems*, H. T. Banks, Ed., SIAM Publ., Philad., 1990

On the positive square root of the fourth derivative operator, Quarterly of Applied Mathematics, 46 (1988), pp. 751 - 773.

Spectral and asymptotic properties of linear elastic systems with internal damping, Proc. Conf. on Boundary Stabilization and Control of Systems Governed by PDE's, Clermont-Ferrand, 1988

(with T. P. Svobodny) *Phase identification in linear time-periodic systems*, IEEE Transactions, Vol. AC 34 (1989), pp. 218-220

(With G. Chen et al) *Analysis, designs and behavior of dissipative joints for coupled beams*, SIAM J. Appl. Math. 49 (1989), pp. 1665-1693

Computational study of the Korteweg - de Vries equation with localized control action, in "*Distributed Parameter Control*

Systems: New Trends and Applications", G. Chen, E. B. Lee, W. Littman and L. Markus, Eds., Marcel Dekker, Inc., New York, 1990, pp. 195 - 203

Approximation of input-output operators for distributed parameter systems, Proc. 1990 IEEE-SIAM Conf. on Dec. & Cont., Honolulu, December 1990.

A comparison of certain elastic dissipation mechanisms via decoupling and projection techniques, to appear in Quarterly of Applied Mathematics, 1991

Neutral FDE canonical representations of hyperbolic systems, Jour. Int. Eq'ns. 3 (1991), 129 - 166

A general framework for elastic systems with indirect damping, submitted to J. Math. Anal. Appl.

(With G. Weiss) *A general necessary condition for exact observability*, submitted to SIAM J. Cont. Opt.

(With B.-Y. Zhang) *Controllability and stabilizability of the third order linear dispersion equation on a periodic domain*, to appear.

Papers by Dr. Scott W. Hansen

Frequency - Proportional Damping Mechanisms in Elastic Beams, Doctoral thesis, University of Wisconsin-Madison, Dec., 1988

(with G. Weiss) *The Operator Carleson Measure Criterion for Admissibility of Control Operators for Diagonal Semigroups on* ℓ^2. Systems and Control Letters, 1991

Papers by Prof. Guenter Leugering

A Decomposition Method for Integro-Partial Differential Equations and Applications. to appear in Math. Pures et Appl., 1991

Papers by Dr. George Weiss.

Transfer Functions of Regular Systems: Part I, submitted to Trans. of the AMS,

Robustness of Feedback Systems with Respect to Small Time Delays, Proc. Conf. on Dec. & Contr., Honolulu, Dec., 1990

Two Conjectures on the Admissibility of Control Operators, Proc. Vorau Symposium on Ident. & Contr. of Distr. Systems, July, 1990, to be published by Birkhauser.

Representations of Shift Invariant Operators on ℓ^2 by Transfer Functions in H^∞: An Elementary Proof, a Generalization to ℓ^p and a Counterexample for ℓ^∞, Math. for Control, Signals and Systems, 1991

5. Partially Supported Visitors and Speakers

Speakers partially supported by the grant during the reporting period with the general area of their presentation:

Prof. Katherine Kime, Case - Western Reserve University
Control theory of the Schrödinger Equation

Prof. Joseph Watkins, University of Southern California
Semigroups and Stochastic Processes

Prof. Luther White, University of Oklahoma
Identification of Coefficients in Elliptic Systems

Prof. B. - Y. Zhang, University of Cincinnati
Control Theory of the Korteweg - de Vries Equation

Prof. Jack Hale, Georgia Tech
Periodic solutions of infinite dimensional nonlinear systems.

Prof. Walter Rudin, University of Wisconsin
Some aspects of the theory of several complex variables.

Prof. G. Chen, Texas A. & M. University
Control of wave equations with localized controls.

Prof. W. Hereman, Colorado School of Mines
Analytic methods for analysis of nonlinear PDEs with solitary wave Solutions

Prof. V. Komornik, University of Bordeaux
Minimal time controllability of the wave equation with boundary controls

Prof. L. F. Ho, Wright State University
Control of the Euler - Bernoulli beam equation with localized controls

6. Travel to Scientific Meetings Partially Supported by AFOSR 89-0031

Grant funds were used during the reporting period to attend and present lectures at the following meetings. In each case a talk was presented describing AFOSR - supported research.

Conference on Decision and Control, Tampa, FL., December, 1989

SIAM Meeting on Control in the 1990's, San Francisco, May, 1989

Three Rivers Applied Mathematics Symposium, Pittsburgh, PA, April, 1990

SIAM Summer Meeting, Chicago, Illinois, June 1990

Regional AMS Meeting, Tampa, Florida, March, 1991

(Travel by V. Komornik) Conference on Decision and Control, Honolulu, December, 1990

(Travel by G. Weiss) SIAM Meeting, San Francisco, Feb. 1991

APPENDIX:

Some Remarks on Experimental Determination of Modal Damping Rates in Elastic Beams*

by

David L. Russell
Department of Mathematics, Virginia Tech.

* Supported in part by AFOSR Grant 89 - 0031

Some Remarks on Experimental Determination of Modal Damping Rates in Elastic Beams

by

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1. Background Let us begin with a protective disclaimer of all intention to treat our subject in any degree of completeness, particularly in regard to giving an account of historical efforts to determine damping rates in elastic systems. A very fine account of the latter is provided by Bell in [1], particularly in regard to the nineteenth century studies of Kelvin, Voigt and Weber. An overview of recent studies in this direction is given in Bert's article [2]. We should also duly note the very significant study [12] of Zener in which he presents a variety of theoretical explanations accounting for internal energy losses in elastic systems.

The projected use of extended elastic structures in a variety of space applications has led, during recent decades, to a significant revival of interest in the study of internal damping mechanisms in such structures. The research so motivated has been greatly facilitated by modern methods of data collection and analysis, such as laser vibrometry and Fast Fourier Transform (FFT) techniques allowing far more comprehensive studies of damping than were possible in earlier times, particularly in the high frequency range. Even more recently, as illustrated by some vibration problems encountered with the Hubble space telescope, it has become ever more evident that such studies are vital if future space platforms are to function satisfactorily. In manned space stations damping is likely to be important in all frequency ranges, from the lower frequencies due to excitation of natural modes of vibration by movement of masses within the structure, docking maneuvers, etc., to high frequency vibrations excited by rotating electrical machinery, "popping" of joints during thermal expansion or contraction, sharp impacts, and many other causes.

Earthbound laboratory studies of internal damping in elastic structures has been complicated by the very real difficulty of eliminating energy losses to the external environment, which are often much greater than the internal losses which are the subject of study. This is particularly true in the high frequency range where vibration of the structure excites acoustic waves in the atmosphere and the support platform for the experiment. Three main sources of external losses may be cited: i) losses due to attachment of sensing devices such as accelerometers, strain gauges, etc., ii) losses due to excitation of sound waves in the atmosphere and atmospheric viscosity, and iii) losses due to interaction of the sample with its supporting structure, such as a clamp, e.g.. The first of these can be largely eliminated with the use of laser vibrometer measurements. Ironically, the use of that technique aggravates the problems associated with the other two because laser vibrometry is a sensitive process; it is usually difficult to obtain a satisfactory reflected signal from the sample unless the sample is supported in such a way as to minimize rotation. This problem leads to the necessity of using, in the case of samples suspended by thin nylon filaments ("pseudo-free" configuations), fairly large samples with significant rotational inertia. In the case of lighter samples it becomes necessary to study the sample in a "cantilever" configuration using a rather massive clamp. In either case, experience indicates significant energy losses through the stretched filaments or the supporting clamp. Laser vibrometry also complicates the use of vacuum chambers to reduce atmospheric effects since a laser beam often cannot be transmitted through the glass windows of such a chamber without significant interference. Thus the vibrometer needs to be in the chamber with the sample with attendant difficulties in aiming the laser and adjustment of the associated electro-optic devices.

In the article to follow we present the results of the writer's experience over the last several years in attempting to answer two questions about internal damping in elastic beams. These questions are the following.

i) What do laboratory experiments indicate concerning the qualitative character of the functional relationship between modal damping rates arising from internal energy dissipation and the corresponding modal frequencies?

ii) What precautions, both in the original experiment and in the

subsequent mathematical analysis of the experiment can be taken to minimize the influence of external energy losses so that the observed damping rates may reasonably be attributed to internal dissipation?

A preliminary exploration of both of these questions appears in [10]; it amounts to an informal account of our understanding of the issues and the implications of our experimental data at the time when that article was written. Developments since then indicate that our early experiments were more seriously contaminated by the effects of external losses than we believed at that time. Also in the interim a better understanding has been gained with regard to the mathematical modelling of both internal and external energy losses in elastic beams and of mathematical procedures which can be used, in conjunction with corresponding experimental studies, to minimize the effects of the latter.

In §2 we discuss, from a mathematical viewpoint, what losses may be expected from a vibrating beam to supporting filaments or to the surrounding atmosphere. Since these losses appear to be relatively modest, particularly at high beam vibration frequencies, we indicate at the end of that article the nature of our best measurements of internal damping in certain types of beams suspended by thin nylon filaments.

In the article [10] we discussed a mathematical approach to determining the magnitude of energy losses from a beam to a supporting clamp, holding the beam in the so-called "cantilever" configuration. There we represented the clamping device as a much thicker beam to which the thin beam under study is coupled in a particular way. In §3 we present a revised analysis of this model confirming the results obtained in a more *ad hoc* manner in that article.

Finally, in §4, we discuss an experimental / mathematical procedure based on observation of modal shapes of a beam subjected to sinusoidal forcing which shows promise of being able to identify internal damping rates in a manner minimally affected by losses to a supporting clamp. We exhibit experimental and computational data plots to indicate the promise inherent in this approach.

2. Losses to Supporting Filaments and the Atmosphere

As we have indicated in §1, one of the ways in which one may attempt to minimize external energy losses in beam damping experiments is to support the beam by long thin filaments, commonly nylon fishing line, as shown in Figure 2.1 at the end of this section. In our earlier article [10] we provided an analysis of energy losses from the beam to these supporting filaments. Here we will indicate the framework used in that analysis and restate the conclusions, referring the reader to the original paper for the details of the estimates.

Since we are interested in energy losses from the elastic beam to the supporting filament, we will use the energy conservative Euler - Bernoulli equation to model the beam. Assuming the beam to have length L, uniform linear mass density m and bending modulus EI, all constant, and assuming the length of the filament supporting the beam to be ℓ and the linear mass density to be r, we have as the approximate energy expression for the system

$$\mathcal{E} = \frac{1}{2}\int_0^L \left[m \left(\frac{\partial w}{\partial t}\right)^2 + EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right] dx + \frac{1}{2}\, gmL \left(\frac{v(0,t)}{\ell}\right)^2$$

$$+ \frac{1}{2}\int_0^{\ell} \left[r \left(\frac{\partial v}{\partial t}\right)^2 + mL \left(\frac{\partial v}{\partial s}\right)^2 \right] ds \;.$$

with the constraint $w(L,t) = v(0,t)$. Here w is the beam deflection as a function of its longitudinal parameter x and v is the filament deflection as a function of its longitudinal parameter s. The last term in the first line comes from lifting of the beam as it swings on the supporting filament. From this energy form one readily derives the equations of motion

$$m \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0, \quad 0 \leq x \leq L \;, \tag{2.01}$$

$$r \frac{\partial^2 w}{\partial t^2} - \rho L \frac{\partial^2 w}{\partial s^2} = 0, \quad 0 \leq s \leq \ell \;, \tag{2.02}$$

and the boundary conditions

$$\frac{\partial^2 w}{\partial x^2}(0,t) = 0, \quad \frac{\partial^3 w}{\partial x^3}(0,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(L,t) = 0 , \tag{2.03}$$

$$EI\frac{\partial^3 w}{\partial x^3}(L,t) + mL\,\frac{\partial v}{\partial s}(0,t) + \frac{gmL}{\ell}\,w(L,t) = 0 . \tag{2.04}$$

We assume that waves propagating through the filament are not reflected at $x = \ell$; this corresponds to the boundary condition

$$\frac{\partial v}{\partial t}(\ell,t) + \left(\frac{mL}{r}\right)^{1/2} \frac{\partial v}{\partial s}(\ell,t) = 0 . \tag{2.05}$$

It is not supposed that the last condition is completely realistic. It simply replaces more complicated assumptions ensuring that very little, if any, energy entering the the supporting filament, or string, is reflected back to the beam. We are assuming that waves in the string move only in the direction from $s = 0$ to $s = \ell$. Then, in fact, equation (2.05) is valid with ℓ replaced by s, $0 \leq s \leq \ell$; in particular it is true at $s = 0$. Using this in (2.04) along with the constraint $w(L,t) = v(0,t)$ we arrive at

$$EI\,\frac{\partial^3 w}{\partial x^3}(L,t) - (rmL)^{1/2}\,\frac{\partial w}{\partial t}(L,t) + \frac{gmL}{\ell}\,w(L,t) . \tag{2.06}$$

A straightforward spectral analysis, similar to that appearing in the article [9], shows that the resulting modal damping exponents are asymptotically constant with the limiting value

$$\frac{2EI}{\rho L}(rmL)^{1/2} .$$

It is easy to see that the attachment of n parallel filaments corresponds to replacing r by nr. Thus, as we vary the number n of supporting filaments, the damping rate should be asymptotically proportional to $n^{1/2}$. It is clear from this that by supporting the same beam in several different configurations with different numbers of identical filaments it is, at least in principle, possible to identify the energy loss rates to the supporting filaments and subtract them from the overall energy loss rates to provide an improved estimate of the

internal dissipation in the beam.

Let us now turn our attention to the question of losses to the surrounding atmosphere due to generation of acoustic waves from the lateral surfaces of the beam. It must be recognized that there may be other atmospheric effects than these; for example, viscosity effects, particularly due to airflow around the edges and free ends of the vibrating beam. These may be expected to have very complicated effects and we make no attempt to account for these in the present discussion. In order to keep our treatment uncomplicated, and yet fairly realistic, let us consider the configuration shown in Figure 2.2; the ends $x = 0$ and $x = L$ of the beam are identified, so that the resulting structure is L-periodic in the x direction. For high frequency beam vibrations, which are our main interest, acoustic waves are generated predominantly in the direction orthogonal to the lateral faces of the beam. Hence we study the linearized acoustic equations in the strip $\mathbf{S}$: $-\infty < y < \infty$, $0 \leq x < L$, with $x = 0$ again identified with $x = L$. A moment of consideration leads one to conclude that atmospheric density variations induced by motion of the beam will be odd functions of y; hence we may confine our study to the half strip corresponding to $y \geq 0$. We begin with the Euler equations for compressible flow in two dimensions, as given in [6] (cf. p. 600 ff.), for example:

$$\rho \frac{\partial u}{\partial t} + \rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y} + p' \frac{\partial \rho}{\partial x} = 0 , \tag{2.07}$$

$$\rho \frac{\partial v}{\partial t} + \rho\, u \frac{\partial v}{\partial x} + \rho\, v \frac{\partial v}{\partial y} + p' \frac{\partial \rho}{\partial y} = 0 , \tag{2.08}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) = 0 , \tag{2.09}$$

where ρ is the atmospheric density, u,v are, respectively, the fluid velocity components in the x,y directions, and $p = p(\rho)$ describes the constitutive relation between pressure and density. To obtain linearized equations for small amplitude waves we let ρ_0 be the mean atmospheric density and we represent ρ in the form $\rho = \rho_0 + \varepsilon\, \delta\rho$, $u = \varepsilon\, \mu$, $v = \varepsilon\, \nu$. Setting $\mathfrak{K} = p'(\rho_0)$ and collecting terms of first order in ε in (2.07) - (2.09), finally multiplying the last equation by $\mathfrak{K}/\rho_0$, we arrive at the linearized equations

$$\rho_0 \frac{\partial \mu}{\partial t} + \mathfrak{R} \frac{\partial \delta\rho}{\partial x} = 0 \ , \tag{2.10}$$

$$\rho_0 \frac{\partial \nu}{\partial t} + \mathfrak{R} \frac{\partial \delta\rho}{\partial y} = 0 \ , \tag{2.11}$$

$$(\mathfrak{R}/\rho_0) \frac{\partial \delta\rho}{\partial t} + \mathfrak{R} \left[\frac{\partial \mu}{\partial x} + \frac{\partial \nu}{\partial x} \right] = 0 \ . \tag{2.12}$$

This is a linear symmetric hyperbolic system with conserved energy integral

$$\mathcal{E} = \frac{1}{2} \iint_{R} \left[\rho_0 \mu^2 + \rho_0 \nu^2 + (\mathfrak{R}/\rho_0)\ \delta\rho^2 \right] dx\ dy \ , \tag{2.13}$$

defined (minimally) on any bounded region R contained in the half strip described above.

Almost any linear operation on the variables μ, ν, $\delta\rho$ will produce a solution z of the wave equation

$$\frac{\partial^2 z}{\partial t^2} - \mathfrak{R} \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right] = 0 \ ; \tag{2.14}$$

the trick is to find the linear operation for which the energy associated with (2.14), i.e.,

$$E = \frac{1}{2} \iint_{R} \left\{ \left(\frac{\partial z}{\partial t} \right)^2 + \mathfrak{R} \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] \right\} dx\ dy \ , \tag{2.15}$$

agrees with the physical energy (2.13). Assuming the mean value of each of μ, ν, $\delta\rho$ to be zero, we may define the inverse Laplace operator Δ^{-1} on each of these quantities or their derivatives. Applying $\partial^2/\partial t \partial x$ to (2.10) and $\partial^2/\partial t \partial y$ to (2.11) and adding, we have

$$\rho_0 \frac{\partial^2}{\partial t^2} \mathrm{div}(\mu, \nu) + \mathfrak{R} \frac{\partial}{\partial t} \Delta \delta\rho = 0 \ .$$

Applying Δ^{-1} we have, with

$$z = \Delta^{-1}\mathrm{div}(\mu,\nu), \tag{2.16}$$

$$\rho_0 \frac{\partial^2 z}{\partial t^2} + \mathfrak{K}\frac{\partial \delta\rho}{\partial t} = \rho_0 \frac{\partial^2 z}{\partial t^2} - \rho_0 \mathfrak{K}\ \mathrm{div}(\mu,\nu) = \rho_0\left[\frac{\partial^2 z}{\partial t^2} - \mathfrak{K}\ \Delta z\right] = 0$$

and we conclude that (2.13) is satisfied with z given by (2.16). Applying $\partial/\partial x$ to (2.10) and $\partial/\partial y$ to (2.11), adding and then applying Δ^{-1}, we have

$$\frac{\partial z}{\partial t} = -\frac{\mathfrak{K}}{\rho_0}\delta\rho \ . \tag{2.17}$$

Using this equation to solve for $\delta\rho$ in (2.10), (2.11) we have

$$\frac{\partial}{\partial t}\left[\mu - \frac{\partial z}{\partial x}\right] = 0 \ , \ \frac{\partial}{\partial t}\left[\nu - \frac{\partial z}{\partial y}\right] = 0 \ .$$

From this it can be shown, much as in [8], that

$$\mu = \frac{\partial z}{\partial x} \ , \quad \nu = \frac{\partial z}{\partial y} \ .$$

Then (2.13) becomes

$$\mathcal{E} = \frac{1}{2}\iint_{\mathfrak{R}} \left[\rho_0\left[\frac{\partial z}{\partial x}\right]^2 + \rho_0\left[\frac{\partial z}{\partial y}\right]^2 + (\rho_0/\mathfrak{K})\left[\frac{\partial z}{\partial t}\right]^2\right] dx\ dy = \frac{\rho_0}{\mathfrak{K}}\ \mathbf{E}$$

and we see that z, as given by (2.16), yields equivalent energy integrals; indeed, replacing z by $(\mathfrak{K}/\rho_0)^{1/2}z$ the energy integrals agree.

Accordingly the total energy expression for the coupled beam - atmosphere system becomes

$$\mathcal{E} = \frac{1}{2}\int_0^L \left[m\left(\frac{\partial w}{\partial t}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] dx + \frac{\rho_0}{2\mathfrak{K}}\iint_{S}\left[\left(\frac{\partial z}{\partial t}\right)^2 + \mathfrak{K}\left(\frac{\partial z}{\partial x}\right)^2 + \mathfrak{K}\left(\frac{\partial z}{\partial y}\right)^2\right] dxdy.$$

Differentiating with respect to t, assuming the overall system to be energy conservative, and recognizing that the force due to the pressure variation across the beam leads to the equation

$$m \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + 2\varepsilon\, \delta\rho = 0 \ ,$$

the usual calculations lead to the interface condition

$$0 = \int_0^L \left[\varepsilon \frac{\partial w}{\partial t}(x,t)\ \delta\rho(x,0,t) + \rho_0 \frac{\partial z}{\partial y}(x,0,t) \frac{\partial z}{\partial t}(x,0,t)\right] dx =$$

$$\int_0^L \left[\varepsilon \frac{\partial w}{\partial t}(x,t) - \kappa \frac{\partial z}{\partial y}(x,0,t)\right] \delta\rho(x,0,t) \ ,$$

from which we conclude that the appropriate boundary condition for z is

$$\frac{\partial z}{\partial y}(x,0,t) = \frac{\varepsilon}{\kappa} \frac{\partial w}{\partial t}(x,t) \ . \tag{2.18}$$

Let us, for convenience, normalize L to 2π. We introduce the Fourier expansions

$$w(x,t) = \sum_{k=-\infty}^{\infty} w_k(t) e^{ikx}, \quad z(x,y,t) = \sum_{k=-\infty}^{\infty} z_k(y,t) e^{ikx},$$

$$\delta\rho(x,y,t) = \sum_{k=-\infty}^{\infty} \delta\rho_k(y,t) e^{ikx} \ .$$

Then, with $\alpha = EI/m$, the coefficients $w_k(t)$ satisfy the ordinary differential equations

$$\frac{d^2 w_k}{dt^2} = -k^4 \alpha\, w_k(t) - 2\varepsilon\, \delta\rho_k(0,t), \ -\infty < k < \infty \ , \tag{2.19}$$

while the coefficients $z_k(y,t)$ obey the partial differential equations

$$\frac{\partial^2 z_k}{\partial t^2} - \mathcal{K} \frac{\partial^2 z_k}{\partial y^2} + \mathcal{K} k^2 z_k = 0 , \quad -\infty < k < \infty , \tag{2.20}$$

and the boundary conditions at y = 0 (cf.(2.18))

$$\frac{\partial z_k}{\partial y}(0,t) = \frac{\varepsilon}{\mathcal{K}} \frac{dw_k}{dt}(t) , \quad -\infty < k < \infty . \tag{2.21}$$

Using (2.17) we may replace (2.19) by

$$\frac{d^2 w_k}{dt^2} + k^4 \alpha w_k(t) - \frac{2\varepsilon\rho_0}{\mathcal{K}} \frac{\partial z_k}{\partial t}(0,t) = 0, \quad -\infty < k < \infty . \tag{2.22}$$

In order to be able to use (2.18) to replace $\frac{\partial z_k}{\partial t}$ in (2.22) by $\frac{dw_k}{dt}$, we need to establish a connection between the t and y derivatives of z_k at $y = 0$. To this end we need a more detailed description of the forced oscillations in the atmosphere. To this end we attempt a solution of (2.20) - (2.22) in the form

$$z_k(y,t) = \zeta_k \, e^{\lambda_k t} \, e^{\mu_k y} , \quad w_k(t) = \omega_k \, e^{\lambda_k t} . \tag{2.23}$$

Then from (2.20) we see quite directly that

$$\lambda_k^2 = K \mu_k^2 - k^2 , \tag{2.24}$$

while (2.21) and (2.22) combine to give

$$\begin{bmatrix} \lambda_k^2 + \alpha k^4 & -2\varepsilon\rho_0 K^{-1}\lambda_k \\ \varepsilon K^{-1}\lambda_k & -\mu_k \end{bmatrix} \begin{bmatrix} \omega_k \\ \zeta_k \end{bmatrix} = 0 .$$

Setting the determinant of the matrix equal to zero and substituting (2.24), we have the polynomial equation in μ_k

$$K \mu_k^3 - 2\varepsilon\rho_0 K^{-1} \mu_k^2 + (\alpha k^4 - k^2) \mu_k + 2\varepsilon^2\rho_0 K^{-2} k^2 = 0 \ . \quad (2.25)$$

Let

$$\mu_k = \tilde{\mu}_k \ k^2 \ . \quad (2.26)$$

Substituting (2.26) into (2.25) and dividing by $\tilde{\mu}_k k^6$, we have, as $k \to \infty$,

$$K \ \tilde{\mu}_k^2 - 2\varepsilon^2\rho_0 K^{-1} k^{-2} \tilde{\mu}_k + \alpha + \mathcal{O}(k^{-2}) = 0 \ ,$$

which gives

$$\tilde{\mu}_k = \varepsilon^2\rho_0 K^{-2} k^{-2} \pm i \ (\alpha K^{-1})^{1/2} + \mathcal{O}(k^{-2})$$

and thus

$$\mu_k = \pm \ i \ (\alpha K^{-1})^{1/2} k^2 + \varepsilon^2\rho_0 K^{-2} + \mathcal{O}(k^{-2}) \ . \quad (2.27)$$

Substituting (2.27) into (2.24) we find that

$$\lambda_k / \mu_k = \pm \ K^{1/2} + \mathcal{O}(k^{-2}), \ k \to \infty \ . \quad (2.28)$$

The - sign is used since it corresponds to waves moving away from the elastic beam in the positive y direction. Then an easy computation using the assumed form (2.23) and (2.28) shows that

$$\frac{\partial z_k}{\partial t} \ (0,t) = \left[-K^{1/2} + \mathcal{O}(k^{-2})\right] \frac{\partial z_k}{\partial y} \ (0,t) = \left[-\varepsilon K^{-1/2} + \mathcal{O}(k^{-2})\right] \frac{dw_k}{dt} \ (t)$$

and thus (2.22) becomes

$$\frac{d^2 w_k}{dt^2} + \left[2\varepsilon^2\rho_0 K^{-3/2} + \mathcal{O}(k^{-2})\right] \frac{dw_k}{dt} + k^4 \ \alpha \ w_k(t) \ , \ -\infty < k < \infty \ .$$

From these equations we conclude that rate of energy dissipation in the k-th vibrational mode is asymptotically uniform as $k \to \infty$ and

directly proportional to the mean atmospheric density ρ_0.

In Figure 2.3 we show superimposed plots of successive power spectra for an aluminum bar supported by two nylon filaments at normal atmospheric pressure; the delay from the beginning of the first sampled time interval to the beginning of the second is 2.8 seconds. The lighter curve is the power spectrum corresponding to the first interval; its peaks are highlighted with a dark dot. The darker curve corresponds to the second power spectrum. (See [9] for a more detailed account of the manner in which these successive power spectra are obtained.) Figure 2.4 plots the differences of the peak values, corresponding to the damping rate in db/sec for the natural modes of vibration with the indicated frequencies. These plots very strongly indicate a rate of damping at least linearly proportional to frequency. The anomalously high damping rates at the lower end of the spectrum are due to losses to the resonant modes of the supporting filaments. Our mathematical analysis indicates that we need not anticipate that the linear character of the damping versus frequency relationship in the high frequency range is seriously corrupted by losses to the supporting filaments or the atmosphere.

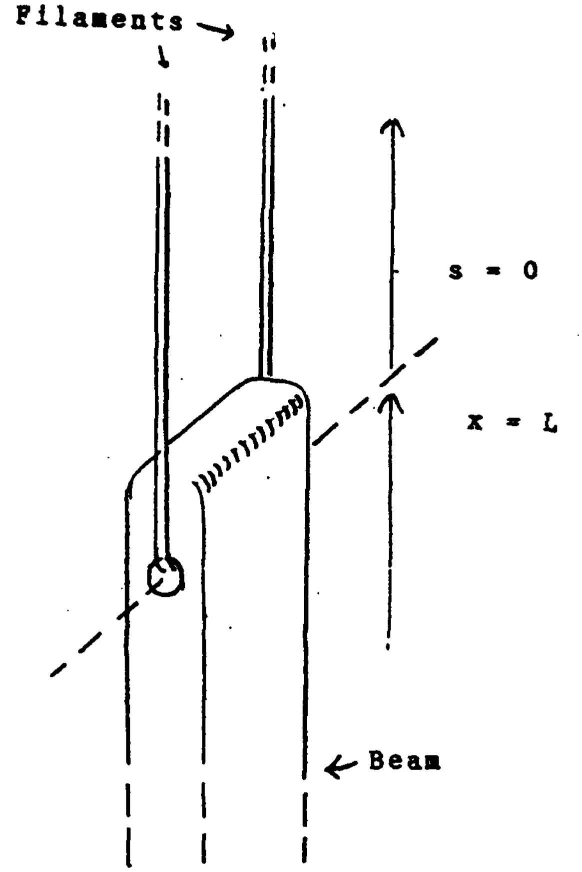

Figure 2.1 "Pseudo-free" beam configuration

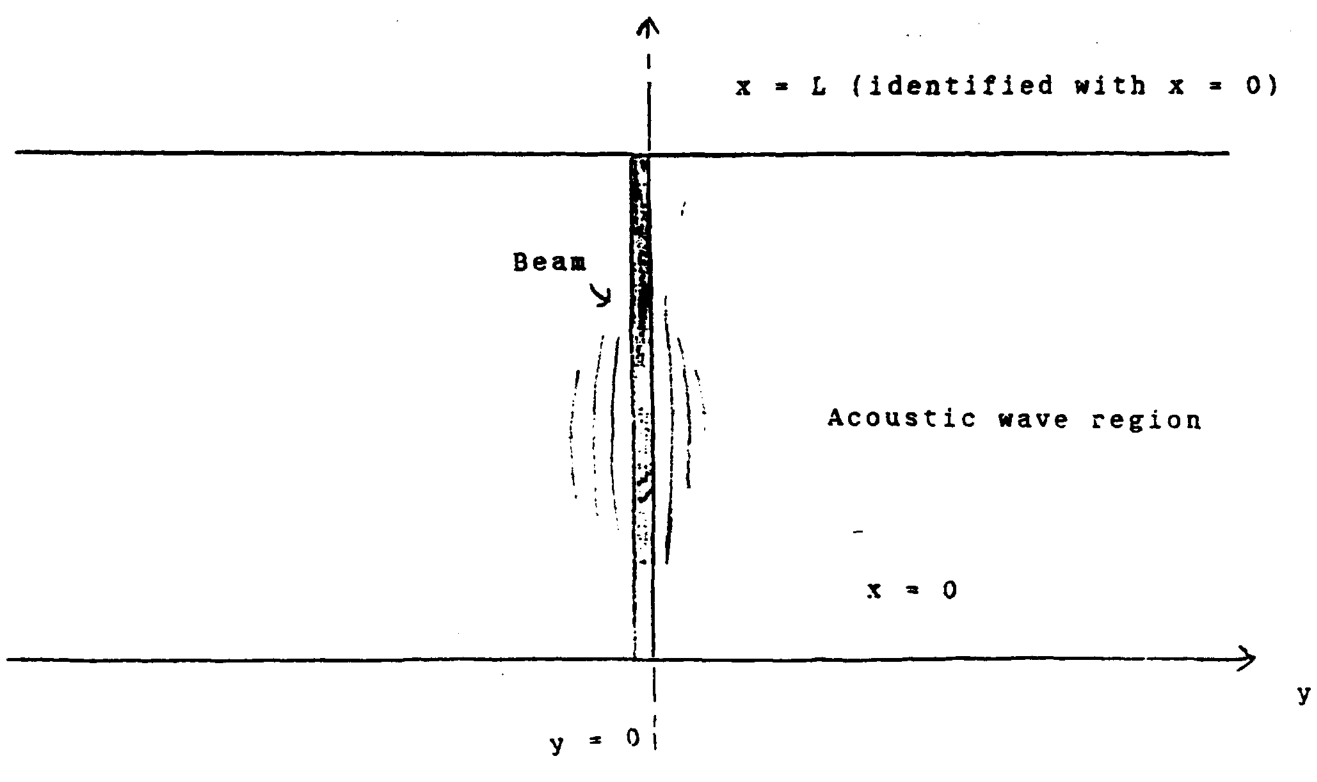

Figure 2.2 Schematic for acoustic energy losses

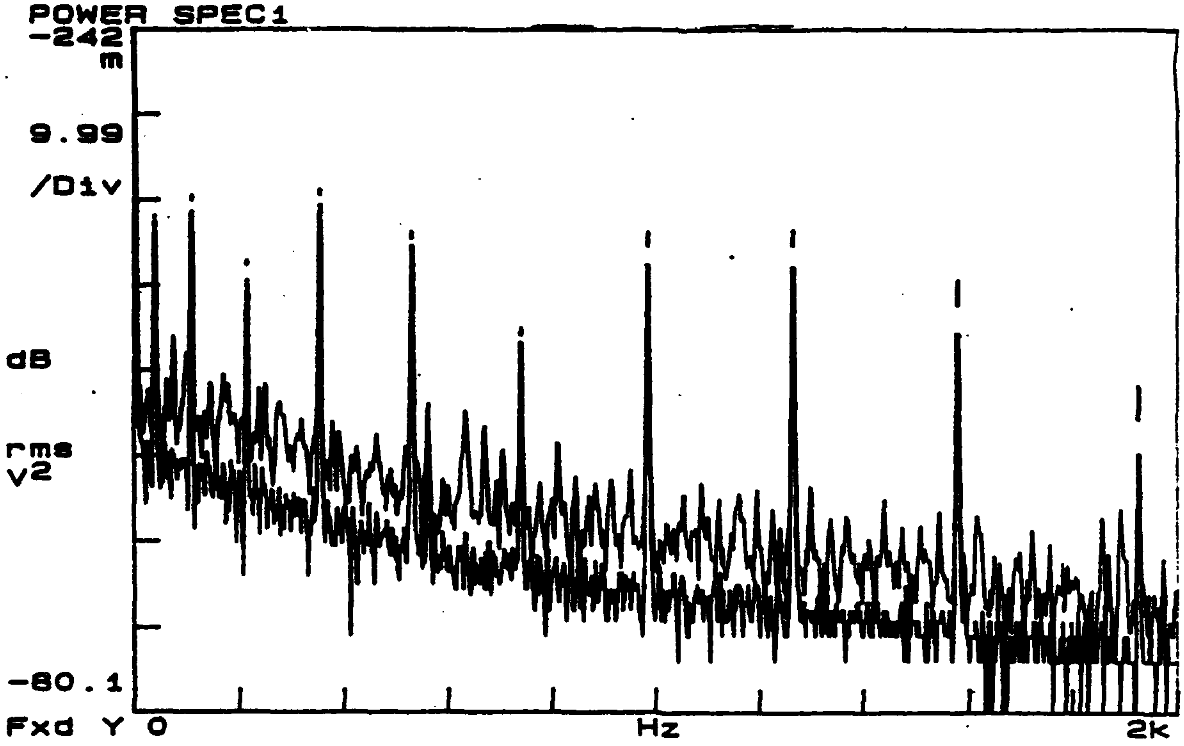

Figure 2.3 Superimposed power spectra; aluminum beam

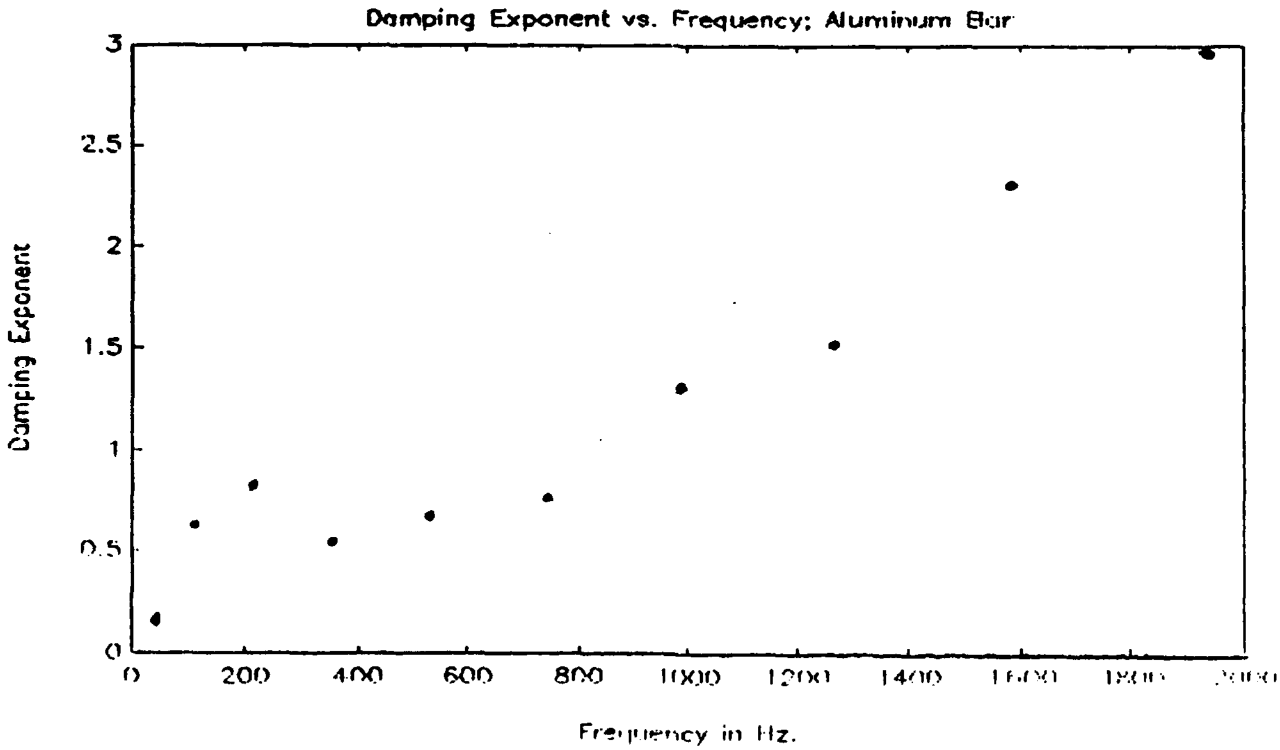

Figure 2.4 Modal damping exponents: aluminum beam

3. External Energy Losses for Clamped Beams

As in our earlier article [10], we consider a thin beam whose lateral displacement is described by $w(x,t)$, $0 \leq x \leq L$. This beam is assumed clamped at $x = 0$ by a device which we model mathematically as a thick beam with displacement $W(x,t)$. We consider "monochromatic", i.e., single frequency, solutions of the combined system under the assumption that "outgoing" waves, moving toward $-\infty$ in the clamping beam are the only motion present in that structure. For definiteness we assume the $x = L$ endpoint of the thin beam to be free but other energy-conservative boundary assumptions would serve equally well. Our analysis here will differ somewhat from that given in [10], where we assumed the thin beam to have infinite length. We refer the reader to that article for explanatory figures.

We treat the thin beam in the Euler - Bernoulli framework with equation of motion

$$m \frac{\partial^2 w}{\partial t^2} + ei \frac{\partial^4 w}{\partial x^4} = 0 , \qquad (3.01)$$

m being the mass density per unit length and ei the bending modulus. At $x = L$ we impose the familiar free endpoint boundary conditions. The thick beam to which the thin beam is attached is modelled via a similar equation with corresponding parameters M and EI:

$$M \frac{\partial^2 W}{\partial t^2} + EI \frac{\partial^4 W}{\partial x^4} = 0 . \qquad (3.02)$$

The clamping assumption yields the kinematic, or essential, boundary conditions

$$w(0,t) = W(0,t), \quad \frac{\partial w}{\partial x}(0,t) = \frac{\partial W}{\partial x}(0,t) . \qquad (3.03)$$

The relevant energy expression is

$$\mathcal{E} = \frac{1}{2} \int_{-\infty}^{0} \left[M \left(\frac{\partial W}{\partial t} \right)^2 + EI \left(\frac{\partial^2 W}{\partial x^2} \right) \right] dx + \frac{1}{2} \int_{0}^{L} \left[m \left(\frac{\partial w}{\partial t} \right)^2 + ei \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dx \qquad (3.04)$$

and it is easy to see that the conditions which should be satisified

for energy to be conserved at the joint x = 0 between the thin and thick beams is

$$EI \frac{\partial^2 W}{\partial x^2}(0,t) = ei \frac{\partial^2 w}{\partial x^2}(0,t) , \tag{3.05}$$

$$EI \frac{\partial^3 W}{\partial x^3}(0,t) = ei \frac{\partial^3 w}{\partial x^3}(0,t) . \tag{3.06}$$

Let us begin the analysis by studying solutions of (3.02) with the special form

$$W(x,t) = e^{\lambda t} e^{\mu x} . \tag{3.07}$$

We further restrict these solutions by insisting that they be "outgoing solutions", i.e., that they should represent waves moving away from x = 0 toward $-\infty$. To facilitate analysis of what that means, let us set

$$\lambda = \alpha\mu \tag{3.08}$$

where

$$\alpha = \alpha_0 + i\ \alpha_1 \tag{3.09}$$

comes from a set of complex numbers yet to be determined. Substituting (3.07), (3.08) into (3.02) and dividing by μ^2 we arrive at the equation

$$M\ \alpha^2 + EI\ \mu^2 = 0$$

so that

$$\mu = \pm \sqrt{M/EI}\ \alpha i \equiv \pm R\ \alpha i . \tag{3.10}$$

Using (3.10) and specializing to the + sign for the present, we have

$$W(x,t) = e^{Ri(\alpha^2 t + \alpha x)} = e^{Ri[(\alpha_0^2-\alpha_1^2)t + \alpha_0 x]}\ e^{[-2\alpha_0\alpha_1 t-\alpha_1 x]} \tag{3.11}$$

The condition for waves to be moving from right to left is that $\alpha_0^2-\alpha_1^2$ and α_0 should have the same sign. Since this motion is supposed to result from a motion in the thin beam decaying in time, both $\alpha_0\alpha_1$ and α_1 in the real exponent should be positive. Thus we are looking for positive numbers α_0 and α_1 with $\alpha_0 > \alpha_1$.

The corresponding displacement w(x,t) of the thin beam will be assumed to have the form (cf. (3.01))

$$w(x,t) = e^{\lambda t}\varphi(x) = e^{i\alpha^2 t}\ \varphi(x)\ ; \tag{3.12}$$

substitution into (3.01) provides the differential equation satisfied by $\varphi(x)$:

$$ei\ \frac{d^4\varphi}{dx^4} = m\alpha^4\varphi\ ,$$

so that, with $r = (m/ei)^{1/4}$,

$$\varphi(x) = c_1\cos r\alpha x + c_2\sin r\alpha x + c_3e^{r\alpha(x-L)} + c_4e^{-r\alpha x}\ ,\ x \in (0,L). \tag{3.13}$$

From the formula (3.07) for W and the identities (3.08), (3.10), we have

$$\frac{\partial^2 W}{\partial t\partial x}(x,t) = \lambda\mu\ e^{\lambda t}e^{\mu x} = -R\alpha^3\ e^{\lambda t}e^{\mu x}\ ,$$

$$\frac{\partial W}{\partial t}\ (x,t) = \lambda\ e^{\lambda t}e^{\mu x} = R\alpha^2 i\ e^{\lambda t}e^{\mu x}\ ,$$

$$EI\ \frac{\partial^2 W}{\partial x^2}\ (x,t) = -\ EI\ R\ \alpha^2\ e^{\lambda t}e^{\mu x}\ ,$$

$$EI\ \frac{\partial^3 W}{\partial x^3}\ (x,t) = -EI\ R^3\alpha^3 i\ e^{\lambda t}e^{\mu x}\ .$$

Then from (3.05), (3.06) and the requirement that the boundary conditions on w at x = 0 should reduce to the free endpoint conditions

as M and EI tend to 0, we obtain

$$ei \frac{\partial^3 w}{\partial x^3}(0,t) = EI \frac{\partial^3 W}{\partial x^3}(0,t) = - EI R^2\alpha \frac{\partial w}{\partial t}(0,t) , \qquad (3.14)$$

$$ei \frac{\partial^2 w}{\partial x^2}(0,t) = EI \frac{\partial^2 w}{\partial x^2}(0,t) = EI \alpha^{-1} \frac{\partial^2 w}{\partial t \partial x} (0,t) . \qquad (3.15)$$

When the boundary conditions (3.14), (3.15) are applied to complex solutions of (3.01) along with the conservative free endpoint conditions

$$\frac{\partial^2 w}{\partial x^2}(L,t) = 0, \quad \frac{\partial^3 w}{\partial x^3}(L,t) = 0 \qquad (3.16)$$

one finds that the time rate of change of

$$E = \frac{1}{2} \int_0^L \left[m \left| \frac{\partial w}{\partial t} \right|^2 + ei \left| \frac{\partial^2 w}{\partial x^2} \right|^2 \right] dx$$

is

$$-\mathrm{Re} \left[ei \frac{\partial^2 w}{\partial x^2}(0,t) \frac{\partial^2 \overline{w}}{\partial t \partial x} (0,t) - ei \frac{\partial^3 w}{\partial x^3}(0,t) \frac{\partial \overline{w}}{\partial t}(0,t) \right] =$$

$$= -\mathrm{Re} \left[ei \frac{\partial^2 w}{\partial x^2}(0,t) \frac{\partial^2 \overline{w}}{\partial t \partial x} (0,t) - ei \frac{\partial^3 w}{\partial x^3}(0,t) \frac{\partial \overline{w}}{\partial t}(0,t) \right] =$$

$$-EI \ \mathrm{Re}(\alpha^{-1}) \left| \frac{\partial^2 w}{\partial t \partial x} (0,t) \right|^2 - EI \ R^2 \ \mathrm{Re}(\alpha) \left| \frac{\partial w}{\partial t}(0,t) \right|^2 \leq 0 \qquad (3.17)$$

with our assumption that $\mathrm{Re}(\alpha) = \alpha_0 > 0$. The remainder of the analysis consists of an examination of the system (3.01), (3.14) - (3.16) with a view to obtaining the high frequency modal damping rates.

From the boundary conditions (3.14), (3.15), (3.16) and from (3.12), setting

$$\gamma = EI/ei, \ \delta = (EI \ R^2)/ei,$$

we obtain for φ the boundary conditions

$$\varphi''(L) = 0, \quad \varphi'''(L) = 0, \tag{3.18}$$

$$\varphi''(0) = \gamma\alpha i\, \varphi'(0), \quad \varphi'''(0) = -\delta\alpha^3 i\, \varphi(0). \tag{3.19}$$

Using the so-called wave propagation method [4], which simplifies eigenvalue approximations by neglecting the exponentially small terms $e^{-r\alpha L}$ arising from substitution of (3.12) into (3.18), (3.19) we obtain the vector equation

$$\begin{bmatrix} -\cos r\alpha L & -\sin r\alpha L & 1 & 0 \\ \sin r\alpha L & -\cos r\alpha L & 1 & 0 \\ -r & -\gamma i & 0 & \gamma i+r \\ \delta i & -r^3 & & -r^3+\delta i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0 . \tag{3.20}$$

Setting the determinant of the indicated matrix equal to zero we have

$$\sin r\alpha L \left[(2\gamma r^3 + 2r\delta)i\right] + \cos r\alpha L \left[2r^4 + 2\gamma\delta + (2\gamma r^3 - 2\delta r)i\right] = 0,$$

or

$$\tan r\alpha L = \frac{2\delta r - 2\gamma r^3 + (2r^4 + 2\gamma\delta)i}{2\gamma r^3 + 2r\delta} .$$

Since $\tan r\alpha L$ behaves like $1/((k+1/2)\pi - r\alpha L)$ in the neighborhood of $(k+1/2)\pi$ for each integer k we have, in first approximation

$$r\alpha L \approx (k+1/2)\pi - \frac{2\gamma r^3 + 2r\delta}{2\delta r - 2\gamma r^3 + (2r^4 + 2\gamma\delta)i} =$$

$$= (k+1/2)\pi - (2\gamma r^3 + 2r\delta)\, \frac{(2\delta r - 2\gamma r^3) - (2r^4 + 2\gamma\delta)i}{(2\delta r - 2\gamma r^3)^2 + (2r^4 + 2\gamma\delta)^2} ,$$

and we conclude that large values of α do, indeed, have the form (3.09) with α_0 and α_1 positive; α_0 grows like $(k+1/2)\pi/rL + c$ for some real constant c while α_1 tends to a fixed positive value. Accordingly,

$$\lambda = R\alpha^2 i = R\left[(\alpha_0^2 - \alpha_1^2) + 2\alpha_0\alpha_1 i\right]i = -2R\alpha_0\alpha_1 + R(\alpha_0^2 - \alpha_1^2)i$$

has negative real part $\approx -2R\alpha_0\alpha_1$ asymptotically proportional to the square root of the frequency $\approx R(\alpha_0^2 - \alpha_1^2)$ as the latter tends to $+\infty$. The other values of λ being complex conjugates of these, we conclude that, granted the validity of this type of clamping model, the damping rate should be asymptotically proportional to the square root of the frequency. This agrees with the result obtained in [10] via slightly different reasoning.

There may be some disagreement that the thin beam/thick beam representation of a clamping device is an accurate one. The assumption of infinite length for the thick beam is certainly suspect as is the assumption that no energy is lost in the clamped joint itself. A somewhat different model is obtained if we abandon the thick beam representation of the clamp and suppose the beam attached at $x = 0$ to a clamping device which may undergo some rotational deformation in the process of beam oscillation and may dissipate energy in that process. For this model we suppose an overall energy functional

$$\mathcal{E}\left(w(\cdot,t),\frac{\partial w}{\partial t}(\cdot,t)\right) = \frac{1}{2}\int_0^L \left[\rho\left(\frac{\partial w}{\partial t}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]dx + \frac{1}{2}\,\mathbf{R}\left(\frac{\partial w}{\partial x}(0,t)\right)^2 . \tag{3.21}$$

The last term corresponds to rotational elasticity of the clamp; if the clamp were assumed perfectly rigid one would impose a boundary condition $\partial w/\partial x(0,t) = 0$. For smooth solutions we may then compute, taking the boundary conditions at $x = L$ to be (3.16) and employing the usual integrations by parts in the second term of the integral,

$$\frac{d\mathcal{E}}{dt} = -EI\frac{\partial^2 w}{\partial x^2}(0,t)\frac{\partial^2 w}{\partial t\partial x}(0,t) + EI\frac{\partial^3 w}{\partial x^3}(0,t)\frac{\partial w}{\partial t}(0,t) + \mathbf{R}\frac{\partial w}{\partial x}(0,t)\frac{\partial^2 w}{\partial t\partial x}(0,t).$$

Assuming

$$w(0,t) \equiv 0, \tag{3.22}$$

the first and third terms reduce to

$$\frac{\partial^2 w}{\partial t \partial x}(0,t)\left[\mathbb{R}\,\frac{\partial w}{\partial x}(0,t) - EI\,\frac{\partial^2 w}{\partial x^2}(0,t)\right] ,$$

which becomes the boundary dissipation term

$$-\sigma\left(\frac{\partial^2 w}{\partial t \partial x}(0,t)\right)^2 \leq 0 \tag{3.23}$$

if we impose at $x = 0$ a second boundary condition

$$\mathbb{R}\,\frac{\partial w}{\partial x}(0,t) - EI\,\frac{\partial^2 w}{\partial x^2}(0,t) + \sigma\,\frac{\partial^2 w}{\partial t \partial x}(0,t) = 0 , \tag{3.24}$$

wherein the last term corresponds to an assumed frictional resistance to rotation in the supporting clamp.

Analysis of the system consisting of the equation (3.01) along with the boundary conditions (3.16), (3.22) and (3.24) by the wave propagation method referred to earlier shows that in this case the rate of energy decay in each mode is proportional to the modal frequency. If this model, or even the model discussed above, for energy loss to a supporting clamp is a valid one it is clear that the determination of internal damping rates in clamped beams from just the "raw" spectral data is a questionable undertaking. In the final section of this article we indicate mathematical procedures whereby the clamping effects can be very nearly eliminated through observation of modal shapes in situations where the supporting clamp, and thus the beam with it, can be forced to oscillate at the corresponding modal frequency.

In view of the above results it is significant that the writer has never been able to obtain satisfactory damping data from clamped metallic beams in the laboratory, notwithstanding any earlier conclusions in [10]. We have obtained fairly good data for composite beams under these circumstances because they have such high internal damping rates that those rates are not unduly masked by losses to the clamping device.

4. Characteristic Features of Elastic Beams in Forced Oscillatory Motion and their Relation to Internal Damping Rates

We provide here a mathematical analysis showing that an elastic beam excited into vibration by sinusoidal motion of its clamping mechanism, exhibits in the resulting beam vibration certain features which are determined by the internal damping rate and are largely independent of the characteristics of the clamp. In Figure 4.1 at the end of this section we indicate, schematically, the experimental set-up and the heuristic basis for the proposed method.

The forced motion of the clamped end of the beam generates waves propagating "outward" toward the free end of the beam, where they are reflected and travel back toward the clamped end. In making this double transit of the beam the waves are, to a greater or lesser extent, depending on the degree of internal damping, attenuated; the work done by the forcing function at the clamped end is, in the periodic steady state situation, just enough to make up for the energy lost in the propagation process together with losses to the supporting structure.

In the loss-free situation, once a periodic motion has been initiated, it continues without any motion of the clamped end. The waves returning from the free end have the same amplitude as the outgoing waves and, at the natural frequencies of the beam, the outgoing and incoming waves interfere constructively to produce a standing wave which is the corresponding natural mode of vibration. When we have internal damping, however, the returning wave has smaller amplitude than the outgoing wave and the constructive interference is not perfect. In particular, as we will see, there are no true "nodes" and the steady state amplitude is not the absolute value of a sinusoidal function of x (plus the small correction due to the real exponential parts of the eigenfunction in question) but, rather, is modified so as to decrease in amplitude as x varies from the clamped end, $x = 0$, to the free end, $x = L$. We will see that this spatial variation in amplitude provides a measure of the internal damping in the beam which is independent of energy losses in the clamping/forcing structure holding the beam at $x = 0$.

In our present mathematical discussion we will assume that the internal damping operator commutes with the elasticity operator. This is not the case for many recent beam dissipation models (cf.[9,10])

but it is true for viscous damping, Kelvin-Voigt damping (cf. [10]) and square-root damping [3,5,7]; it will serve for the present analysis. Accordingly we consider a uniform elastic beam modelled by an an Euler - Bernoulli type equation

$$\rho \frac{\partial^2 w}{\partial t^2} + G \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} = 0, \quad t \in (-\infty,\infty) \ , \quad x \in [0,L] \ , \tag{4.01}$$

where G is a non-negative, in general unbounded, self-adjoint operator commuting with the operator $\partial^4/\partial x^4$. The beam is assumed free at the end $x = L$, corresponding to boundary conditions (3.16). At $x = 0$ we suppose that there is forced motion:

$$w(0,t) = f(t) \ . \tag{4.02}$$

This motion is produced by a corresponding motion of a clamping device. We model the clamp as in the last paragraphs of the preceding section, supposing the overall energy functional to be (3.21). The rate of change of energy is now seen to be

$$\frac{d\mathcal{E}}{dt} = -\left(\frac{\partial w}{\partial t}(\cdot,t),\ G\frac{\partial w}{\partial t}(\cdot,t)\right)_{L^2[0,L]} - EI\frac{\partial^2 w}{\partial x^2}(0,t)\ \frac{\partial^2 w}{\partial t\partial x}(0,t)$$

$$+ EI\frac{\partial^3 w}{\partial x^3}(0,t)\ \frac{\partial w}{\partial t}(0,t) + \mathbb{R}\frac{\partial w}{\partial x}(0,t)\ \frac{\partial^2 w}{\partial t\partial x}(0,t) \ . \tag{4.03}$$

The first term on the right hand side is non-positive and represents internal energy dissipation. From (1.03) the third term is

$$EI\frac{\partial^3 w}{\partial x^3}(0,t)\ \dot{f}(t) \ , \tag{4.04}$$

representing the work done on the beam by the external forcing. With the boundary conditions (3 1) and (4.02) applying at $x = 0$ the total energy dissipation form is ow seen to be

$$-\left[\left(\frac{\partial w}{\partial t}(\cdot,t),\ G\frac{\partial w}{\partial t}(\cdot,t)\right)_{L^2[0,L]} + \sigma\left(\frac{\partial^2 w}{\partial t\partial x}(0,t)\right)^2\right] \leq 0 \tag{4.05}$$

which is balanced by (4.04) over each period of the steady state for-

ced oscillation.

Now we suppose that the function f(t) in (4.02) takes the form

$$f(t) = e^{i\omega t} \tag{4.06}$$

and that the corresponding response takes the form

$$w(x,t) = e^{i\omega t}\,\hat{w}(x,\omega)\ ; \tag{4.07}$$

we will separate out the real component of the motion at the end of our calculations. The boundary conditions for $\hat{w}$ now take the form (cf.(4.02), (3.16), (3.24))

$$\hat{w}(0,\omega) = 1\ ,\ (R + \sigma i\omega)\,\frac{\partial\hat{w}}{\partial x}(0,\omega) - EI\,\frac{\partial^2 w}{\partial x^2}(0,\omega) = 0\ , \tag{4.08}$$

$$\frac{\partial^2\hat{w}}{\partial x^2}(L,\omega) = \frac{\partial^3\hat{w}}{\partial x^3}(L,\omega) = 0\ . \tag{4.09}$$

From our commutativity and non-negativity assumptions on the operator G, we may assume that there is a non-negative function $\gamma(\omega)$, the form of which is to be identified, such that $\hat{w}(x,\omega)$ satisfies the differential equation

$$EI\,\frac{d^4\hat{w}}{dx^4}(x,\omega) + (i\omega\gamma(\omega) - \rho\omega^2)\,\hat{w}(x,\omega) = 0\ . \tag{4.10}$$

Suppressing the role of ω as an argument and setting

$$r = \rho/EI,\quad \omega g = \gamma/EI,\quad \omega^2(r-ig) = \nu^4 e^{4i(\pi/2-\varepsilon)}\ , \tag{4.11}$$

where

$$\nu^4 = \omega^2\sqrt{r^2+g^2}\ ,\quad e^{4i(\pi/2-\varepsilon)} = (r-ig)/\sqrt{r^2+g^2}\ ,$$

the fourth roots of $\omega^2(r-ig)$ take the form

$$\mu_k = \nu\, e^{i(k\pi/2-\varepsilon)} = \nu\, e_k\ ,\quad k = 1,2,3,4\ . \tag{4.12}$$

The indexing is such that μ_k lies in the k-th quadrant of the complex plane in each case, assuming g sufficiently small so that $0 \leqq \varepsilon < \frac{\pi}{2}$.

Accordingly, for some coefficients c_k, k = 1,2,3,4, we have

$$\hat{w}(x,\omega) = c_1\, e^{\mu_1 x} + c_2\, e^{\mu_2 x} + c_3\, e^{\mu_3 x} + c_4 e^{\mu_4 x}\ . \tag{4.13}$$

Applying the boundary conditions (4.08), (4.09) we have the system of equations

$$c_1 + c_2 + c_3 + c_4 = 1\ , \tag{4.14}$$

$$R_1 c_1 + R_2 c_2 + R_3 c_3 + R_4 c_4 = 0\ , \tag{4.15}$$

$$\mu_1^2 e^{\mu_1 L} c_1 + \mu_2^2 e^{\mu_2 L} c_2 + \mu_3^2 e^{\mu_3 L} c_3 + \mu_4^2 e^{\mu_4 L} c_4 = 0\ , \tag{4.16}$$

$$\mu_1^3 e^{\mu_1 L} c_1 + \mu_2^3 e^{\mu_2 L} c_2 + \mu_3^3 e^{\mu_3 L} c_3 + \mu_4^3 e^{\mu_4 L} c_4 = 0\ , \tag{4.17}$$

where, in agreement with (4.08),

$$R_k = (R+\sigma i\omega)\ \mu_k - EI\ \mu_k^2\ ,\quad k = 1,2,3,4\ .$$

The solution of this four dimensional system is rather complicated; it will serve our purposes here to develop an approximate solution, valid for large ν and small ε, again using the wave propagation method. Under these circumstances μ_4 has large positive real part and μ_2 has large negative real part. Assuming c_1 and c_3 can be shown to have relatively moderate values, we omit the second term from equations (4.16) and (4.17). Multiplying the modified equation (4.16) by μ_1 and subtracting the modified equation (4.17) we have

$$c_3 e^{\mu_3 L}(\mu_1\mu_3^2 - \mu_3^3) + c_4 e^{\mu_4 L}(\mu_1\mu_4^2 - \mu_4^3) = 0\ . \tag{4.18}$$

From (4.18) we infer that c_4 must be very small. Omitting the fourth term from equations (4.14), (4.15), we obtain two modified equations. Then, multiplying the modified equation (4.16) by μ_4 and subtracting the modified equation (4.17) we obtain a third equation. These together form a three dimensional system

$$c_1 + c_2 + c_3 = 1 , \tag{4.19}$$

$$R_1c_1 + R_2c_2 + R_3c_3 = 0 , \tag{4.20}$$

$$e^{\mu_1 L}(\mu_4\mu_1^2 - \mu_1^3)c_1 + e^{\mu_3 L}(\mu_4\mu_3^2 - \mu_3^3)c_3 = 0 . \tag{4.21}$$

Solving for c_2 in (4.19) and substituting in (4.20) and (4.21) we obtain the two dimensional system consisting of (4.21) and

$$(R_2-R_1)c_1 + (R_2-R_3)c_3 = R_2 . \tag{4.22}$$

Applying Cramer's rule to (4.21),(4.22) we have

$$c_1 = R_2e^{\mu_3 L}(\mu_4\mu_3^2 - \mu_3^3)/\mathfrak{D}(\mu) , \tag{4.23}$$

$$c_3 = -R_2e^{\mu_1 L}(\mu_4\mu_1^2 - \mu_1^3)/\mathfrak{D}(\mu) , \tag{4.24}$$

where $\mathfrak{D}(\mu)$ is the determinant

$$\mathfrak{D}(\mu) = (R_2-R_1)e^{\mu_3 L}(\mu_4\mu_3^2 - \mu_3^3) - (R_2-R_3)e^{\mu_1 L}(\mu_4\mu_1^2 - \mu_1^3) . \tag{4.25}$$

From (4.23), (4.24), noting that $\mu_3 = -\mu_1$, we see that there is a complex $A = A(R,\sigma,\mu)$ such that (cf.(4.12))

$$c_1 = A\,(e_4 - e_3)\,e^{\mu_3 L},\quad c_3 = A\,(e_1 - e_4)\,e^{\mu_1 L} . \tag{4.26}$$

From (4.18) we also have, since $\mu_3^2 = -\mu_4^2$,

$$c_4 = -\frac{e^{\mu_3 L}(\mu_1\mu_3^2 - \mu_3^3)}{e^{\mu_4 L}(\mu_1\mu_4^2 - \mu_4^3)}\, c_3 = \frac{e_1 - e_3}{e_1 - e_4}\, e^{\mu_3 L} e^{-\mu_4 L}\, c_3 \,. \qquad (4.27)$$

In this approximation $e^{-\mu_4 L} = e^{\mu_2 L}$ is to be regarded as replaceable by zero, unless multiplied by a term comparable to $e^{\mu_4 L}$. Thus we have, using (4.19),

$$c_2 = 1 - c_1 - c_3 = 1 - A\left[e_1(e^{\mu_1 L} + e^{-\mu_1 L}) - e_4(e^{\mu_1 L} - e^{-\mu_1 L})\right]. \quad (4.28)$$

Thus the forced beam oscillation has the approximate form

$$w(x,t) = Ae^{-i\varepsilon}e^{i\omega t}\Big[(1+i)e^{\mu_1(x-L)} + (i-1)e^{-\mu_1(x-L)} + (1-i)e^{i\varepsilon}e^{-\mu_1 L}e^{\mu_4(x-L)}$$

$$+ e^{i\varepsilon}\left[A^{-1} - \left[e_1(e^{\mu_1 L} + e^{-\mu_1 L}) - e_4(e^{\mu_1 L} - e^{-\mu_1 L})\right]\right]e^{-\mu_4 x}\Big] \,. \quad (4.29)$$

Since A depends on σ, the damping parameter of the clamp, the last term also depends on σ. But, since $e^{-\mu_4 x}$ is a factor in this term and since $-\mu_4$ has large negative real part for large values of ω, this term will not be important, again for large ω, at any substantial distance from the clamp. The third term will not be important at any substantial distance from the tip of the beam. Assuming we make measurements in a "central region" of the beam, we can concentrate on

$$w(x,t) \approx Ae^{-i\varepsilon}e^{i\omega t}\left[(1+i)e^{\mu_1(x-L)} + (i-1)e^{-\mu_1(x-L)}\right] \,. \qquad (4.30)$$

We still have A, which depends on σ, as an amplitude factor but, since it multiplies both terms, it will play no role in comparing relative amplitudes at different points x in the central region. (The imaginary part of A simply gives rise to a time phase shift.) This is the basic idea to be used in identifying γ, the damping parameter for the beam itself, since, as we can see from (4.11), μ_1 is determined by the coefficients ρ, EI, and γ appearing in the

equation (4.10). So we now consider just the expression

$$\tilde{w}(x,t) = e^{i\omega t}\left[(1+i)e^{\mu_1(x-L)}+(i-1)e^{-\mu_1(x-L)}\right] =$$

$$2(\cos\omega t + i\sin\omega t)\left[\sinh\mu_1(x-L) + i\cosh\mu_1(x-L)\right] . \quad (4.31)$$

Let us set

$$\mathbf{S} = \sin\left(\frac{\pi}{2} - \varepsilon\right) \approx 1 - \frac{\varepsilon^2}{2} , \quad \mathbf{C} = \cos\left(\frac{\pi}{2} - \varepsilon\right) = \sin\varepsilon \approx \varepsilon . \quad (4.32)$$

Then

$$\mu_1 = \nu\, e^{i\left(\frac{\pi}{2} - \varepsilon\right)} = \nu\,(\mathbf{C} + i\mathbf{S}) , \quad (4.33)$$

$$e^{\mu_1 x} = e^{i\nu\mathbf{S}x}e^{\nu\mathbf{C}x} = (\cos\nu\mathbf{S}x + i\sin\nu\mathbf{S}x)\,e^{\nu\mathbf{C}x} . \quad (4.34)$$

From (4.31)and (4.34) we can see that

$$\mathrm{Re}\,\tilde{w}(x,t) = 2\left\{\cos\omega t\cos\nu\mathbf{S}(x-L)\sinh\nu\mathbf{C}(x-L)\right.$$

$$- \sin\omega t\sin\nu\mathbf{S}(x-L)\cosh\nu\mathbf{C}(x-L) - \cos\omega t\sin\nu\mathbf{S}(x-L)\sinh\nu\mathbf{C}(x-L)$$

$$\left.- \sin\omega t\cos\nu\mathbf{S}(x-L)\cosh\nu\mathbf{C}(x-L)\right\} =$$

$$2\sqrt{2}\left[\cos\omega t\cos\left(\nu\mathbf{S}(x-L) + \frac{\pi}{4}\right)\cosh\nu\mathbf{C}(x-L)\right.$$

$$\left.- \sin\omega t\sin\left(\nu\mathbf{S}(x-L) + \frac{\pi}{4}\right)\sinh\nu\mathbf{C}(x-L)\right] =$$

$$2\sqrt{2}\left[\cos\left(\omega t + \nu\mathbf{S}(x-L) + \frac{\pi}{4}\right)e^{\nu\mathbf{C}(x-L)}+\sin\left(\omega t - \nu\mathbf{S}(x-L) - \frac{\pi}{4}\right)e^{-\nu\mathbf{C}(x-L)}\right] . \quad (4.35)$$

This formula expresses the central region motion in terms of an "outgoing" wave (the second term), attenuated like $e^{-\nu\mathbf{C}x}$ as it moves to

the right, plus an "incoming" wave (the first term), attenuated like $e^{\nu \mathbf{C} x}$ as it moves to the left. From the penultimate formula of (4.35) we see that, for given x in the central region, the amplitude of the time oscillation is

$$2\sqrt{2}\left[\cos^2\left(\nu \mathbf{S}(x-L)+\frac{\pi}{4}\right)\cosh^2\nu \mathbf{C}(x-L) + \sin^2\left(\nu \mathbf{S}(x-L)+\frac{\pi}{4}\right)\sinh^2\nu \mathbf{C}(x-L)\right]^{1/2} =$$

$$2\sqrt{2}\left[\cosh^2\nu \mathbf{C}(x-L) - \sin^2\left(\nu \mathbf{S}(x-L)+\frac{\pi}{4}\right)\right]^{1/2} . \qquad (4.36)$$

When γ (hence also $\mathbf{C}$) = 0 the last expression has zeros where

$$\sin\left(\nu \mathbf{S}(x-L)+\frac{\pi}{4}\right) = \pm 1 , \qquad (4.37)$$

corresponding to "nodes" of the standing wave form. When $\gamma > 0$ there are no true nodes; what are left are "quasi-nodes" corresponding to minima of (4.36). The values of these minima will correspond to an amplitude factor times $\cosh^2\nu \mathbf{C}(x-L) - 1$; by comparing these minima as they occur for different values of x in the central region of the beam one may obtain an estimate for $\nu \mathbf{C} \equiv \mathbf{D}$ for the mode under study. From (4.11) we see that, assuming g, and hence ε, small

$$\varepsilon \approx \frac{1}{4}\sin 4\varepsilon = \frac{g}{4\sqrt{r^2+g^2}} \qquad (4.38)$$

and from (4.11), (4.32) we have

$$\mathbf{D} \approx \nu\varepsilon = \omega^{1/2}\left(r^2+g^2\right)^{1/8}\varepsilon \qquad (4.39)$$

so that

$$\gamma = \omega g \approx 4\omega^{1/2}\left(\sqrt{r^2+g^2}\right)^{3/4}\mathbf{D} . \qquad (4.40)$$

From the last formula we see that a *constant* (i.e., *frequency independent*) damping rate, corresponding to γ constant, is indicated by the following relationship between $\mathbf{D}$ and the frequency, ω:

$$\mathfrak{D} \approx \frac{\gamma}{4\omega^{1/2}\left(\sqrt{r^2+g^2}\right)^{3/4}} = \mathcal{O}(\omega^{-1/2}) \,, \tag{4.41}$$

while *frequency proportional* (sometimes called *structural*) damping corresponds to g constant so that

$$\mathfrak{D} \approx \frac{g\omega}{4\omega^{1/2}\left(\sqrt{r^2+g^2}\right)^{3/4}} = \mathcal{O}(\omega^{1/2}) \,. \tag{4.42}$$

According to this theory, then, the presence of structural damping will be indicated by an estimated value of $\mathfrak{D} = \mathfrak{D}(\omega)$ varying in proportion to $\omega^{1/2}$ as ω becomes large.

In Figures 4.2 and 4.3 we show computed values of the amplitude of the time oscillation for two natural modes of a damped Euler-Bernoulli beam (these are computed exactly, rather than by means of the approximation (4.36)), plotted as functions of the longitudinal beam parameter. In Figures 4.4 and 4.5 we show mode shapes taken in the laboratory by scanning a beam in forced vibration by means of a laser vibrometer. The last two figures do indicate that the minima corresponding to what we have called the quasi-nodes are smaller at the right hand (free) end of the beam than at the left hand (clamp driven) end. The experimental data are not of the quality required for reliable estimation of internal damping rates at the present writing but do indicate that improvement of our scanning techniques and equipment may be expected to yield such estimates in the near future.

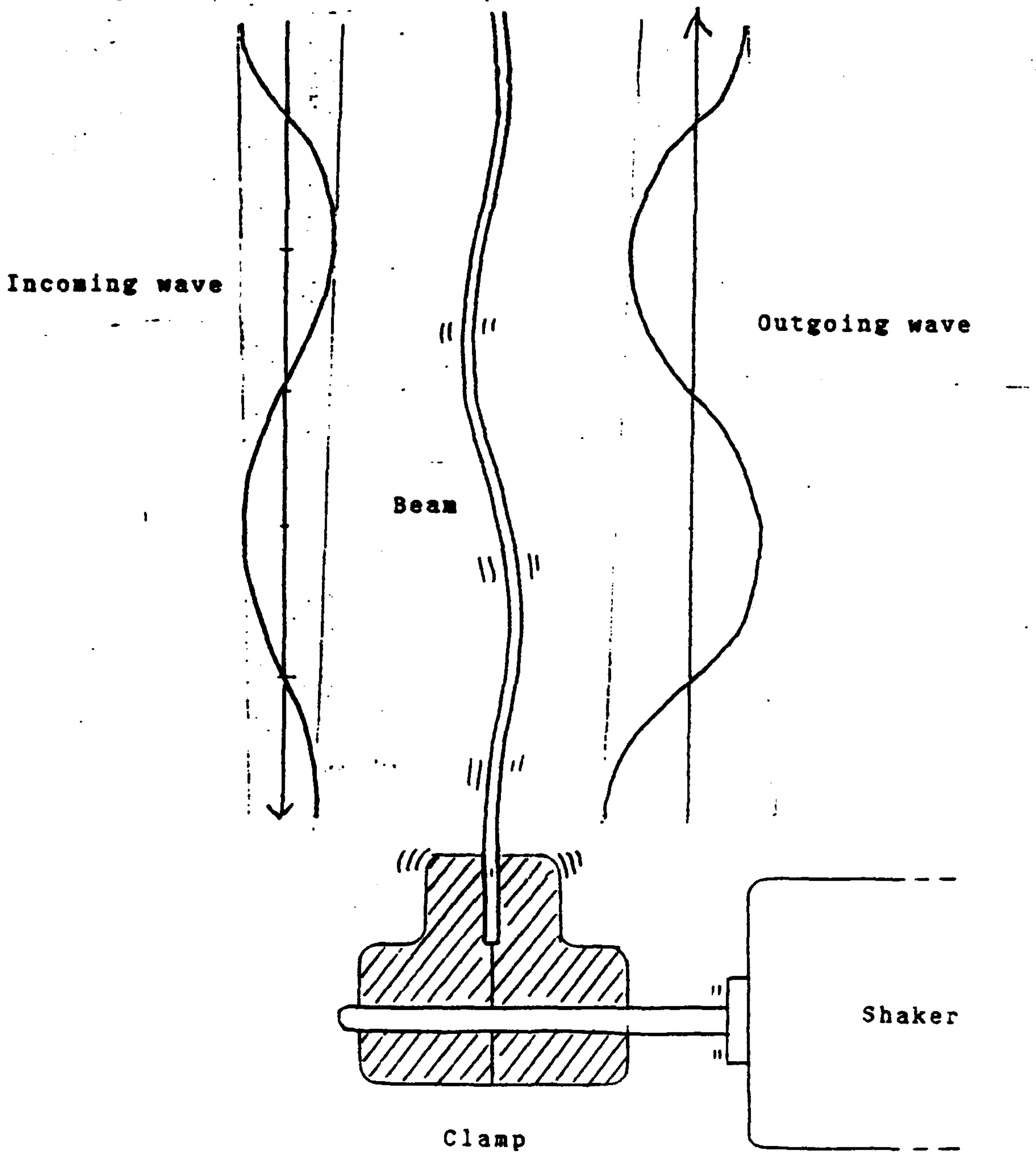

Figure 4.1 Schematic for beam driven in oscillating clamp

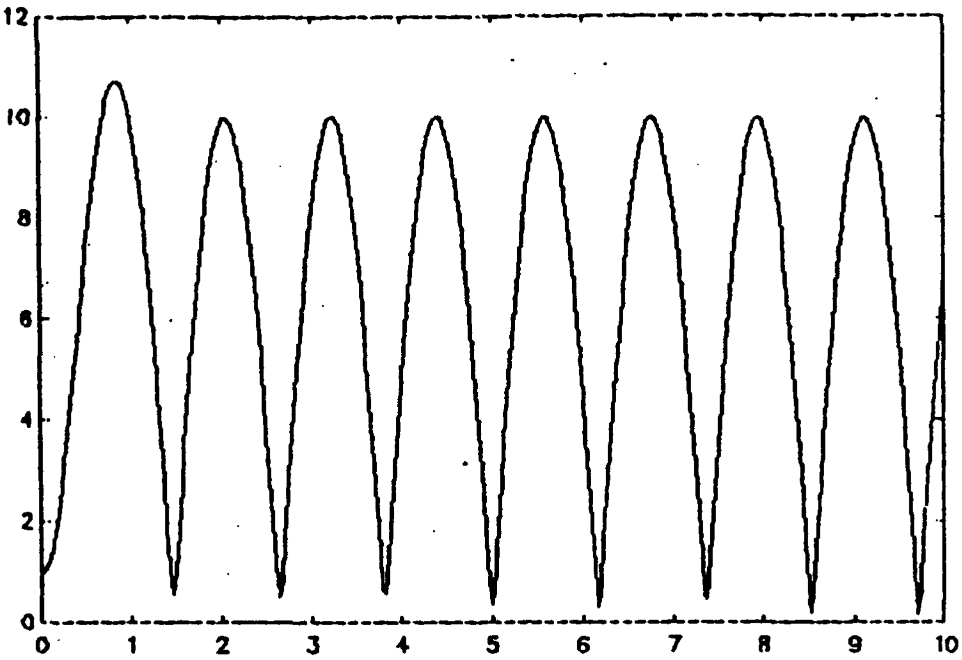

Figure 4.2 Amplitude modulus; driven beam mode

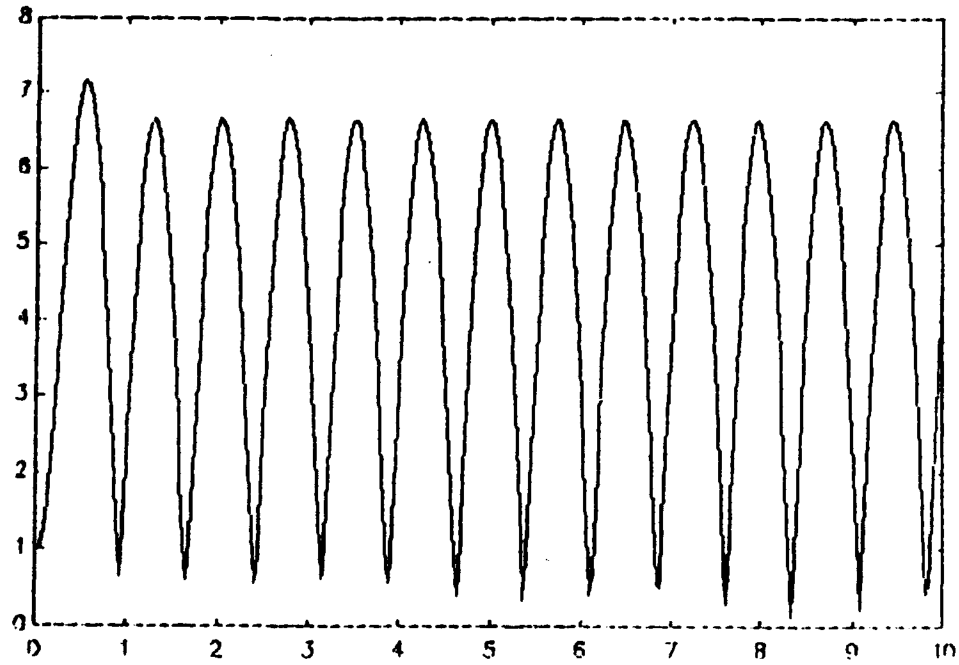

Figure 4.3 Amplitude modulus: driven beam mode

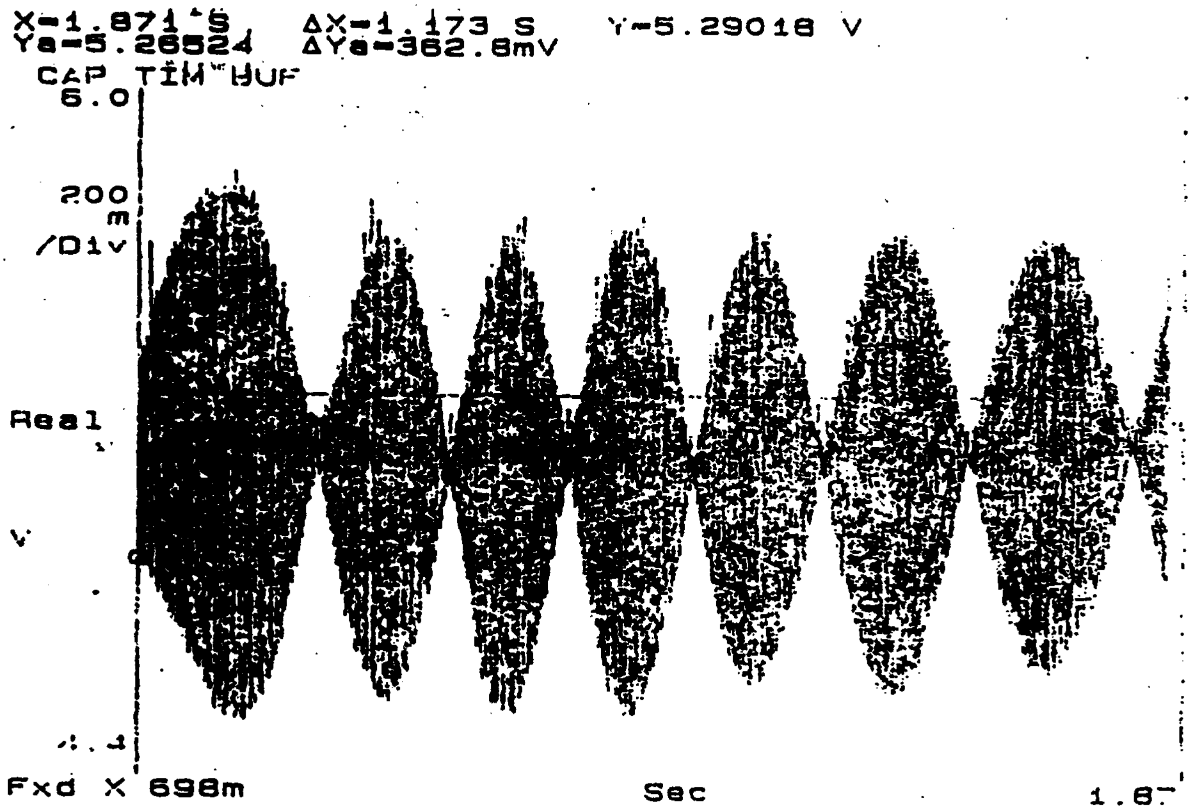

Figure 4.4 Experimental amplitude modulus; aluminum beam

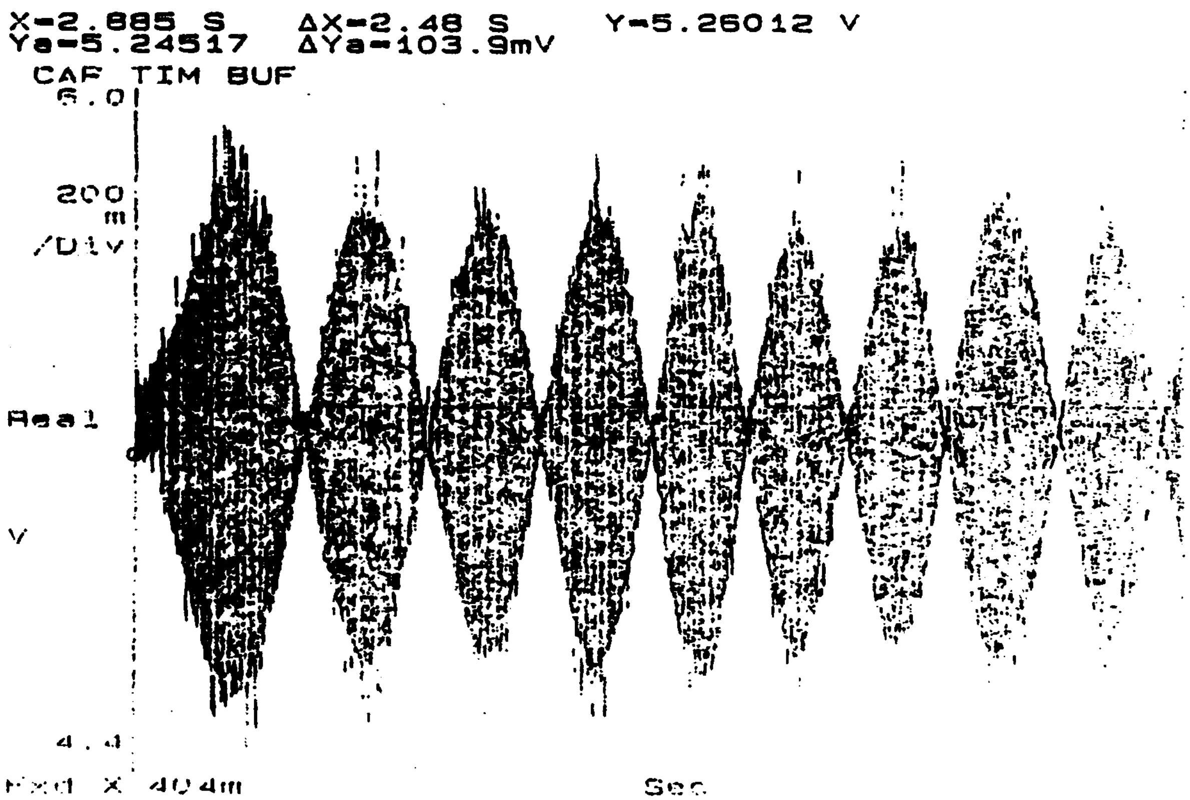

Figure 4.5 Experimental amplitude modulus; aluminum beam

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